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EXACT ANALYTICAL SOLUTION OF A LUMPED MODEL OF THE TRANSIENT CONVECTIVE-RADIATIVE COOLING OF A HOT SPHERICAL BODY IN AN ENVIRONMENT

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Exact Analytical Solution of a Lumped Model of the Transient Convective-Radiative Cooling of a Hot Spherical Body in an Environment

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In this study the exact analytical solution of the lumped parameter model of a nonlinear heat transfer process representing the transient convective-radiative cooling of a spherical body has been obtained. The process is governed by a nonlinear ordinary differential equation, and the exact analytical solution has been found in the implicit form of an elementary transcendental function. The obtained exact analytical solution not only yields accurate results but also successfully simulates a recent experimental study of cooling of metallic ball bearings by the combined mechanism of convection and radiation. In addition, the exact explicit solution for a simplified case of the above problem, recently tackled by several researchers in various approximate ways, has also been found. These exact solutions are quite appealing since they are accurate and superior to the available approximate solutions, provide better insight of the physical process, and can also serve as yardsticks for future testing of the approximate solutions.

Keywords Ball bearings; Conduction; Convection; Exact analytical solution; Mathematical modeling; Radiation

Introduction

Cooling of a body by the combined effect of convection and radiation is widely encountered in many heat transfer operations. Some of the situations where surface radiative and/or convective heat transfer processes play significant role are metallurgical processes, radiation devices in outer space applications, heat transfer from extended surfaces, dynamical thermal behavior of buildings, and cooling of electronic components. For convenience, some of recently carried out studies are outlined in Table I.

Many mathematical modeling approaches are available to portray the above-mentioned unsteady convective-radiative heat transfer processes, starting from the simple lumped parameter model to a more complex distributed parameter model (Siegel and Howell, 1992; Campo and Blotter, 2000; Bejan and Krauss, 2003; Cortés et al., 2003; Modest, 2003; Su, 2004; Liao et al., 2006; Tan et al.,

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Table I. Some recent applications of lumped and distributed models of convective and/or radiative heat transfer from/to a body

Authors	Year	Problem tackled	Model developed	Solution method
Parang, Crocker, and Haynes	1990	Theoretical simulation of the solidification of spherical and cylindrical geometries (groundwater freezing and metal solidification) by convective and radiative cooling	Distributed parameter	Perturbation method
Theodoropoulou, Adomaitis, and Zafiriou	1998	Simulation of an experimental study for determining the spatiotemporal temperature distribution in a wafer during thermal chemical vapor deposition of polysilicon	Distributed parameter	Galerkin's collocation
Campo and Blotter	2000	Simulation of an experimental study of cooling of ball bearing in atmosphere	Lumped parameter	Runge-Kutta-Fehlberg
Bird, Stewart, and Lightfoot	2002	Classical case of the freezing of water in the night sky in deserted areas	Lumped parameter	Numerical solution
Cortés, Campo, and Arauzo	2003	Theoretical comparison of lumped and distributed models for the cooling of plate, cylinder, and sphere	Lumped parameter	Approximate analytical
Su	2004	Comparison of the classical and improved lumped models with the distributed model of a spherical object being cooled by radiation	Lumped and distributed parameter	Approximate analytical for lumped and finite difference method for distributed models

(Continued)

Table I. Continued

Authors	Year	Problem tackled	Model developed	Solution method
Liao, Su, and Chwang	2006	Simulation of cooling of a spherical body	Distributed parameter	Homotopy analysis method
Keshavarz and Taheri	2007	Improved lumped model for convective cooling of several standard geometries	Lumped parameter	Polynomial approximation
Sakin, Ertekin, and Llicali	2009	Simulation of an experimental study of heating process in an oven	Lumped and distributed parameter	Analytical for lumped model and finite difference method for distributed model
Tan, Su, and Su	2009	Comparison of the classical and improved lumped models with the distributed model of a wall being cooled by convection and radiation	Lumped and distributed parameter	Approximate analytical for lumped models and finite difference method for distributed model
Kupiec and Komorowicz	2010	Theoretical study for proposing an improved lumped model for the cooling of a spherical object by radiation	Lumped and distributed parameter	Approximate analytical for lumped model and Crank-Nicolson method for distributed model
Sadat, Dubus, Dez, Tatibouët, and Barrault	2010	Experimental simulation of a dielectric barrier discharge reactor for finding the transient temperature profile after ignition and shutdown	Lumped parameter	Analytical solution

2009; Kupiec and Komorowicz, 2010). The distributed parameter models of these processes are represented by partial differential equations (PDEs) having nonlinear boundary conditions and provide spatial and temporal details of the temperature of the concerned body. However, the numerical solutions of these PDEs are cumbersome and time consuming, especially in repeated calculations. To overcome this difficulty, various attempts have been made to propose different improved lumped parameter models so as to obtain sufficiently accurate information with minimum effort (Su and Cotta, 2001; Cortés et al., 2003; Su, 2004; Keshavarz and Taheri, 2007; Pontedeiro et al., 2008; Tan et al., 2009; Kupiec and Komorowicz, 2010). The equations characterizing the lumped parameter models of these processes are derived by performing some spatial averaging of the concerned PDEs of distributed parameter models and these PDEs are rendered into nonlinear ordinary differential equations (ODEs). In contrast to the distributed parameter model equations, the lumped parameter model equations are mathematically tractable, however, they provide only temporal details of the temperature of the body. Therefore, the choice between these two approaches depends on the degree of accuracy and the level of details required as well as on the effort required to solve these model equations. In convection-radiation processes, the choice between the lumped and distributed parameter models should be made only after properly evaluating the concerned total Biot number $Bi_T (= \frac{h_T l}{k})$, where l is the characteristic dimension of the body and h_T is the total heat transfer coefficient for the combined convection-radiation process ($h_T = h_c + h_r$). It can be noted that the lumped parameter model is valid if $Bi_T < 0.1$ (Campo and Blotter, 2000; Tan et al., 2009).

In this work, efforts have been made to obtain the closed form exact analytical solutions of the lumped parameter models of a nonlinear heat transfer process and one of its simplified cases. The main heat transfer process and its simplified case basically describe the transient heat loss from a spherical body by the collective means of convection and radiation. The governing equations for both situations are given by nonlinear first-order ODEs constituting initial value problems (IVPs) (Siegel and Howell, 1992; Bejan and Krauss, 2003; Modest, 2003; Su, 2004). To the best of the authors' knowledge, a closed form exact analytical solution for either of the model equations is not available in the literature. However, the latter simpler case has recently been investigated by several researchers (Ganji et al., 2007; Rajabi et al., 2007; Domairry and Nadim, 2008) by using different approximate methods, namely PM (perturbation method), HPM (homotopy perturbation method), and HAM (homotopy analysis method), and the solutions were found in terms of some finite series.

In addition, the simulation of a recently conducted experimental case study, representing the transient cooling of a metal ball bearing by the combined mode of convection and radiation (Campo and Blotter, 2000), has also been effectively performed by the derived exact analytical solution of the main heat transfer process.

Mathematical Model of the Cooling of a Spherical Body

For the sake of completeness and brevity, the distributed parameter model for the unsteady-state convective and radiative cooling of a spherical body is presented in this section. The problem is stated as follows: A spherical body having radius R , density ρ , thermal conductivity k , and heat capacity c_p is initially at a higher temperature T_i , and at the outset of the experiment ($t = 0$), it is exposed to an environment at

temperature T_f and average radiation sink temperature T_s . The body is assumed to be homogeneous, isotropic, and opaque, and due to the temperature gradient it starts losing heat energy by the superimposed effects of convection and radiation. For this situation, the governing model equation can be derived by applying the unsteady-state energy balance over a control element in the body and is given by (Su, 2004; Liao et al., 2006):

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(kr^2 \frac{\partial T}{\partial r} \right) \quad (1)$$

The concerned initial condition (IC) and boundary conditions (BC) are:

$$\text{IC} : T = T_i \quad \text{at } t = 0 \quad \forall r \leq R \quad (2a)$$

$$\text{BC I} : -k \frac{\partial T}{\partial r} = h_c(T - T_f) + \sigma \epsilon (T^4 - T_s^4) \quad \text{at } r = R \quad \forall t > 0 \quad (2b)$$

$$\text{BC II} : \frac{\partial T}{\partial r} = 0 \quad \text{at } r = 0 \quad \forall t > 0 \quad (2c)$$

where h_c is the convective heat transfer coefficient, and ϵ and σ are the emissivity of the spherical body and Stefan-Boltzmann constant, respectively. In case of different environment and sink temperatures, it is appropriate to introduce the following adiabatic surface temperature, which simplifies the computational work (Liao et al., 2006):

$$h_c(T_a - T_f) + \sigma \epsilon (T_a^4 - T_s^4) = 0 \quad (3)$$

With the help of the above-defined adiabatic surface temperature (T_a) BC I becomes:

$$\text{BC I} : -k \frac{\partial T}{\partial r} = h_c(T - T_a) + \sigma \epsilon (T^4 - T_a^4) \quad \text{at } r = R \quad \forall t > 0 \quad (4)$$

Now introducing the following dimensionless variables:

$$\theta = \frac{T}{T_i}, \eta = \frac{r}{R}, \theta_a = \frac{T_a}{T_i}, \tau = \frac{\alpha t}{R^2}, Bi_c = \frac{h_c R}{k}, \quad \text{and} \quad N_{rc} = \frac{\sigma \epsilon R T_i^3}{k}$$

Equations (1), (2a), (2c), and (4) can easily be transformed into the dimensionless forms given by Equations (5) and (6a)–(6c). $\alpha (= k/\rho c_p)$ is the thermal diffusivity, Bi_c is the Biot number for convection, and N_{rc} is the conduction-radiation parameter. One should note that the above definition of $Bi_c (= \frac{h_c R}{k})$ is based on the one proposed by Liao et al. (2006), although there are also other definitions of $Bi_c (= \frac{h_c R}{3k})$ (Campo and Blotter, 2000; Tan et al., 2009).

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{\eta^2} \frac{\partial}{\partial \eta} \left(\eta^2 \frac{\partial \theta}{\partial \eta} \right) \quad (5)$$

$$\text{IC} : \theta = 1 \quad \text{at } \tau = 0 \quad \forall \eta \leq 1 \quad (6a)$$

$$\text{BC I : } -\frac{\partial\theta}{\partial\eta} = Bi_c(\theta - \theta_a) + N_{rc}(\theta^4 - \theta_a^4) \quad \text{at } \eta = 1 \forall \tau > 0 \quad (6b)$$

$$\text{BC II : } \frac{\partial\theta}{\partial\eta} = 0 \quad \text{at } \eta = 0 \forall \tau > 0 \quad (6c)$$

For a body with large thermal conductivity and smaller dimensions, the spatial temperature variations in it can be neglected and the lumped parameter model can be used. However, this should be supported by the criteria $Bi_T < 0.3$, where $Bi_T \left(= \frac{(h_c+h_r)R}{k} \right)$ is the overall Biot number and takes into account the Biot numbers for convection and radiation (Campo and Blotter, 2000; Liao et al., 2006; Tan et al., 2009). It should be noted that the definition of Bi_T used in the present study is based on the one proposed by Liao et al. (2006) instead of the one given by Campo and Blotter (2000). Due to this fact the lumped parameter model criteria reduces to $Bi_T < 0.3$ instead of $Bi_T < 0.1$. $h_r \left[= \frac{\sigma\epsilon(T^4 - T_a^4)}{(T - T_a)} \right]$ is the radiative heat transfer coefficient.

For a spherical body, the equation for the lumped parameter model is obtained by using the following definition of the spatially averaged dimensionless temperature (Su, 2004):

$$\theta_{av} = 3 \int_0^1 \theta \eta^2 d\eta \quad (7)$$

The above definition renders the earlier obtained PDE into a first-order nonlinear ODE constituting an IVP. Thus, from Equations (5), (6b), (6c), and (7), one finally gets the following dimensionless equation:

$$\frac{d\theta_{av}}{d\tau} = -3Bi_c(\theta_{av} - \theta_a) - 3N_{rc}(\theta_{av}^4 - \theta_a^4) \quad (8)$$

Similarly, the associated IC, i.e., Equation (6a), with the use of Equation (7), can be expressed in terms of the spatially averaged temperature and attains the following form:

$$\text{IC : } \theta_{av} = 1 \quad \text{at } \tau = 0 \forall \eta \leq 1 \quad (9)$$

Exact Analytical Solutions

In this section, the exact analytical solutions for both cases, i.e., (i) $\theta_a \neq 0$ and (ii) $\theta_a = 0$, are obtained. In addition, an experimental case study is also fruitfully simulated with the help of the exact analytical solution of the general case.

General Case ($\theta_a \neq 0$)

After rearranging Equation (8) and integrating it with the help of Equation (9), one gets:

$$\frac{1}{N_{rc}} \int_1^{\theta_{av}} \frac{d\theta_{av}}{\frac{Bi_c}{N_{rc}}(\theta_{av} - \theta_a) + (\theta_{av}^4 - \theta_a^4)} = -3 \int_0^\tau d\tau \quad (10)$$

The partial fraction decomposition of the left-hand side of Equation (10) yields the following simplified form:

$$\frac{1}{N_{rc}} \left[\int_1^{\theta_{av}} \frac{A_1 d\theta_{av}}{(\theta_{av} - r_1)} + \int_1^{\theta_{av}} \frac{B_1 d\theta_{av}}{(\theta_{av} - r_2)} + \int_1^{\theta_{av}} \frac{(C_1 \theta_{av} + D_1) d\theta_{av}}{(\theta_{av}^2 - (r_3 + r_4)\theta_{av} + r_3 r_4)} \right] = -3 \int_0^\tau d\tau \quad (11)$$

where r_1 , r_2 , r_3 , and r_4 are the roots of the quartic equation $\frac{Bi_c}{N_{rc}}(\theta_{av} - \theta_a) + (\theta_{av}^4 - \theta_a^4) = 0$ and the explicit expressions of these roots are given in Appendix I. From Appendix I it is revealed that r_1 is real and positive, r_2 is real and negative, and r_3 and r_4 are complex conjugates; this fact is true for all the possible combinations of θ_a and $\frac{Bi_c}{N_{rc}}$. A_1 , B_1 , C_1 , and D_1 are the constants appeared during the partial fraction decomposition, and their expressions are given in Appendix II.

Now, expressing the complex conjugate pair r_3 and r_4 as $r_3 = a + ib$ and $r_4 = a - ib$, and integrating Equation (11), one obtains the following equation:

$$\begin{aligned} \frac{1}{N_{rc}} \left[\frac{(D_1 + C_1 a)}{b} \tan^{-1} \left[\frac{\theta_{av} - a}{b} \right] + \frac{1}{2} C_1 \ln[(\theta_{av} - a)^2 + b^2] \right. \\ \left. + A_1 \ln[\theta_{av} - r_1] + B_1 \ln[\theta_{av} - r_2] \right]_1^{\theta_{av}} \\ = -3 \int_0^\tau d\tau \quad (12) \end{aligned}$$

After simplifying the above equation, the following exact analytical solution is found:

$$\begin{aligned} \frac{1}{N_{rc}} \left[\frac{(D_1 + C_1 a)}{b} \left(\tan^{-1} \left[\frac{\theta_{av} - a}{b} \right] - \tan^{-1} \left[\frac{1 - a}{b} \right] \right) \right. \\ \left. + \frac{1}{2} C_1 \ln \left[\frac{(\theta_{av} - a)^2 + b^2}{(1 - a)^2 + b^2} \right] \right. \\ \left. + A_1 \ln \left[\frac{\theta_{av} - r_1}{1 - r_1} \right] + B_1 \ln \left[\frac{\theta_{av} - r_2}{\theta_{av} - r_2} \right] \right] = -3\tau \quad (13) \end{aligned}$$

Using Equation (13), the transient profiles of θ_{av} have been drawn in Figure 1 for various values of Bi_c , N_{rc} , and θ_a , which match exactly with their numerical counterparts and thus validate the above exact analytical solution. It can be verified that the selected values of the parameters (Bi_c , N_{rc} , and θ_a) satisfy the lumped parameter model criteria, i.e., $Bi_T (= Bi_c + Bi_r) < 0.3$.

Moreover, in the case of heating of the spherical body by convection and/or radiation, minor changes have to be made in Equation (2b) such that the constant ambient temperature is greater than the object's temperature ($T_f > T \geq T_i$). The obtained transient temperature profile will be just the reverse of the one shown in Figure 1; in other words, a mirror image showing increasing trend of the temperature profile will be obtained.

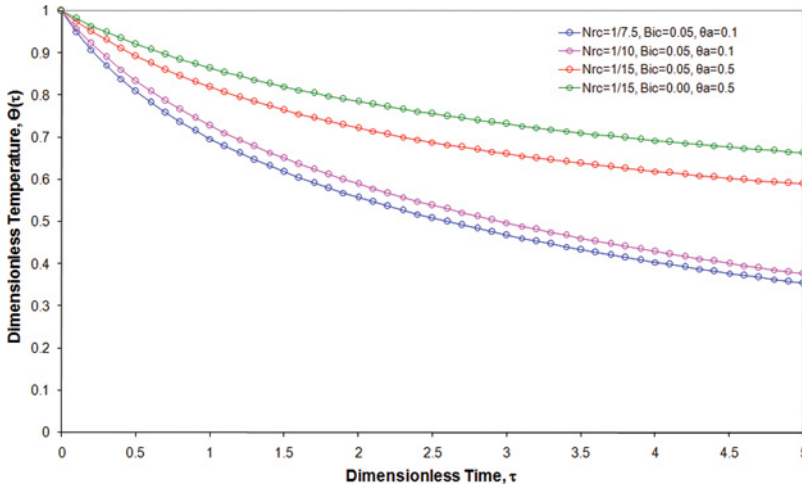


Figure 1. Transient profiles of dimensionless temperature for various values of N_{rc} , Bi_c , and θ_a ; solid lines: exact analytical solution; open circles: numerical solution. (Figure provided in color online.)

Simulation of an Experimental Case Study

In this subsection, the practical applicability of the just obtained exact analytical solution (Equation (13)) has been demonstrated by simulating an existing experimental study of cooling of a metal ball bearing by the modes of convection and radiation (Campo and Blotter, 2000). The use of the lumped parameter model for this experimental study was shown to be justified by Campo and Blotter (2000), i.e., $Bi_T \left[= \frac{(h_c+h_r)R}{k} \right] < 0.3$. Later, we also verified this condition. Therefore, the application of the presently derived exact analytical solution is justified.

Out of the two tests carried out by Campo and Blotter (2000), the data of test 1 corresponding to cooling at a higher temperature have been selected in this work, although the experimental results of test 2 can also be simulated in a similar fashion. Necessary details of test 1 are summarized in Table II. and complete details can be

Table II. Experimental data used for simulation

Variable/Parameter	Notation	Value
Source: Campo and Blotter (2000)		
Initial temperature of ball bearing	T_i	823 K
Constant room air temperature	T_f	302 K
Radiation sink temperature	T_s	302 K
Diameter of ball bearing	D	$0.953 \times 10^{-2} \text{ m}$
Emissivity of ball bearing	ϵ	0.7
Source: Bejan and Kraus (2003)		
Density of ball bearing	ρ	$7865 \text{ kg} \cdot \text{m}^{-3}$
Specific heat of ball bearing	c_p	$460 \text{ J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$
Thermal conductivity of ball bearing	k	$47 \text{ W} \cdot \text{m}^{-1} \cdot \text{K}^{-1}$

found in the original work (Campo and Blotter, 2000). The parameters ρ , c_p , and k were not given by Campo and Blotter (2000), hence, in the present study, their values have been taken from Bejan and Kraus (2003); these values do not affect the results in any significant way and the results remain almost exactly the same.

The heat transfer process involved in the experimental study of Campo and Blotter (2000) is governed by the same model equation, i.e., Equations (8) and (9), however, the convective heat transfer coefficient is now a weak nonlinear function of the temperature and is given by the following relation (Campo and Blotter, 2000):

$$h_c(T) = 9.03 + 2.95(T - 302)^{0.25} \text{ (W/m}^2 \text{ K)} \quad (14)$$

With the introduction of the above expression for convective heat transfer coefficient, the lumped parameter model equation attains the following dimensional form:

$$\rho c_p \frac{R dT}{3 dt} = - \left[9.03 + 2.95(T - 302)^{0.25} \right] (T - T_a) - \sigma \in (T^4 - T_a^4) \quad (15a)$$

$$\text{IC : } T(0) = T_i = 823 \text{ K} \quad (15b)$$

Now, to show the applicability of the derived analytical solution (Equation (13)), the average value of the convective heat transfer coefficient, h_{cav} , over the concerned temperature range ($T_i - T_a$) is used in place of its temperature-dependent form (Equation (14)). The average value of the convective heat transfer coefficient is found to be 20.3051 W/m²K and has been evaluated by using the following relation:

$$h_{cav} = \frac{1}{(T_i - T_a)} \int_{T_a}^{T_i} h_c(T) dT \quad (16)$$

By replacing the temperature-dependent $h_c(T)$ with its average value h_{cav} in Equation (15a), one gets the following equation:

$$\rho c_p \frac{R dT}{3 dt} = -h_{cav}(T - T_a) - \sigma \in (T^4 - T_a^4) \quad (17)$$

Equation (17) is now forced to attain a dimensionless form similar to Equation (8) by using the previously defined dimensionless variables. In doing so, the values of θ_a , Bi_c , and N_{rc} are found to be 0.366950, 0.002059, and 0.002243, respectively. One should note that the value of Bi_c is based on h_{cav} . In addition, the maximum values of h_c and h_r , i.e., at the beginning of the cooling of the bearing, are 23.1239 and 34.3099 W/m² K, respectively, and corresponding to these values, the maximum value of the total Biot number (Bi_T) is 0.005823, which satisfies the lumped parameter model criteria ($Bi_T < 0.3$). Hence, the assumption of the lumped parameter model is valid for whole of the duration of this experimental study.

Now, corresponding to the above values of θ_a , Bi_c , and N_{rc} , the four roots are found to be: $r_1(=\theta_a) = 0.366950$, $r_2 = -1.076538$, $r_3 = 0.354794 + 0.879018i$, and $r_4 = 0.354794 - 0.879018i$. Substituting these values in Equation (13), one obtains

the following equation:

$$\begin{aligned}
 &404.24805 + 183.80592 \tan^{-1}[0.40362 - 1.13763\theta_{av}] + 399.70488 \ln[\theta_{av} - 0.36695] \\
 &\quad - 145.10981 \ln[0.77267 + (\theta_{av} - 0.35479)^2] \\
 &\quad - 109.48525 \ln[\theta_{av} + 1.07654] = -3\tau
 \end{aligned}
 \tag{18}$$

Comparison among Analytical, Numerical, and Experimental Results

A comparison among the analytical, numerical, and experimental results has been made by plotting the respective temperature profiles in Figure 2. The numerical results have been found by numerically solving Equations (15a) and (15b) with the help of the inbuilt command “NDSolve” of Mathematica software. As evident from Figure 2, the numerically obtained profiles depict close agreement with the experimentally obtained profiles of Campo and Blotter (2000) (one should note that some of the experimental readings in our work have been obtained from Figure 2 of Campo and Blotter (2000) with the help of the user-friendly software Plot Digitizer, available free online, as only a few values were tabulated in Table 1 of Campo and Blotter (2000)).

Beside these two temperature profiles, Figure 2 also shows the temperature profiles obtained by using the analytical and numerical solutions of the modified equation (Equation (17)). From this figure, it is clear that a close match exists between these two solutions and thus signifies the correctness of the analytical solution. Moreover, like the numerical results of Equation (15a), the results of Equation (17) obtained by either the analytical solution (Equation (18)) or the numerical method, also match well with the experimental data. This validates the use of the analytical solution and the average convective heat transfer coefficient, h_{cav} . Hence,

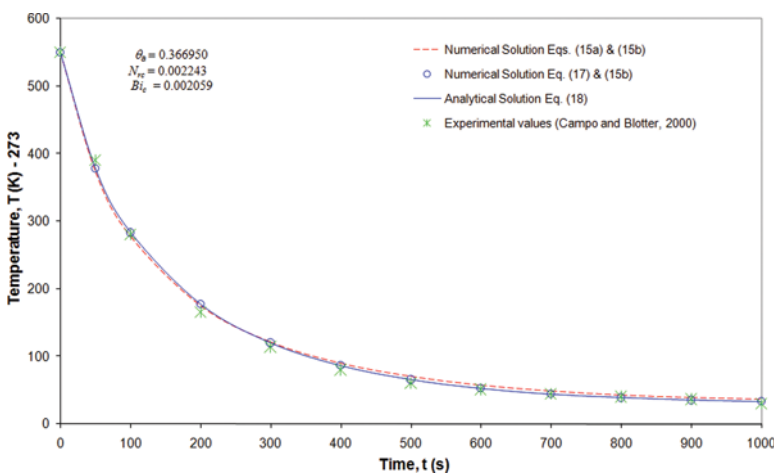


Figure 2. Transient temperature profiles for the cooling of a metal ball bearing by convection and radiation mechanisms ($\theta_a = 0.366950$, $N_{rc} = 0.002243$, $Bi_c = 0.002059$). (Figure provided in color online.)

it can be concluded that no appreciable change in the results is observed if one models the above heat transfer process by using Equation (17) instead of Equation (15a).

Simplified Case: ($\theta_a = 0$)

This particular situation is characterized by the fact that both the surroundings and the sink temperatures are the same and equal to zero, i.e., $T_s = T_f = 0$. From Equation (3) this implies that $T_a = \theta_a = 0$. This situation may arise in outer space or in a vacuum. The model equation of this specific situation has been solved by several researchers by using various approximate methods, e.g., PM, HPM, and HAM (Ganji et al., 2007; Rajabi et al., 2007; Domairry and Nadim, 2008). However, the model equation as considered by these researchers contains a term corresponding to convection and is therefore invalid, since at $\theta_a = 0$ (absolute zero temperature of the surroundings), there will be no convective heat transfer, i.e., $Bi_c = 0$. Due to this reason we have not considered their model equation, rather, we have presented the appropriate model equation for this situation ($\theta_a = 0$). This model equation is given below in dimensionless form along with the associated IC (it should be noted that as opposed to the model equation considered by Ganji et al. (2007), Rajabi et al. (2007), and Domairry and Nadim (2008), the following model equation does not contain the convective heat transfer term because $Bi_c = 0$):

$$\frac{d\theta_{av}}{d\tau} + 3N_{rc}\theta_{av}^4 = 0 \quad (19a)$$

$$\text{IC: } \theta_{av}(\tau = 0) = 1 \quad (19b)$$

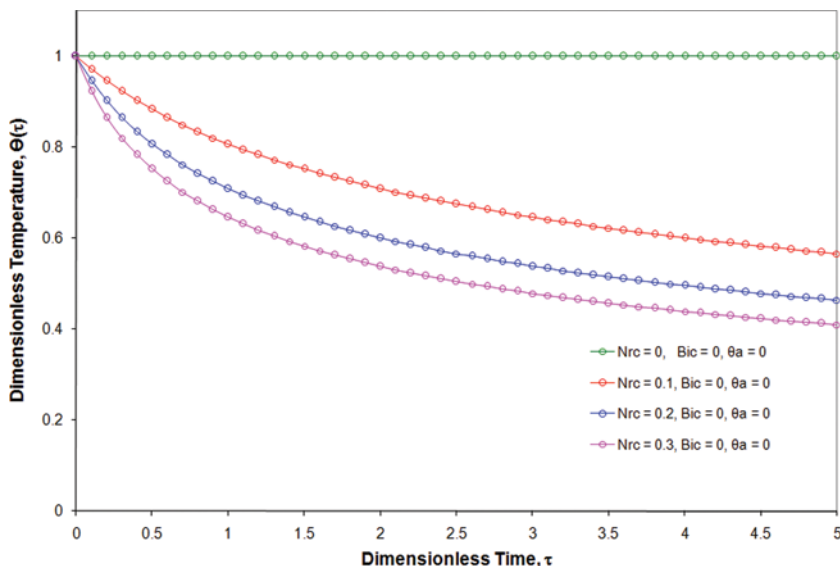


Figure 3. Transient profiles of dimensionless temperature at $Bi_c = 0$ and $\theta_a = 0$ for various values of N_{rc} ; solid lines: exact analytical solution; open circles: numerical solution. (Figure provided in color online.)

Integrating the above equation after separating the variables and using IC, one finds the following exact analytical solution:

$$\theta_{av} = \frac{1}{(1 + 9N_{rc}\tau)^{1/3}} \quad (20)$$

Comparison between Analytical and Numerical Results

For the several values of parameter N_{rc} ($= 0, 0.1, 0.2, 0.3$), the dimensionless temperature profiles obtained by using the exact analytical solution (Equation (20)) and the numerical solution are plotted in Figure 3. These values of N_{rc} satisfy the lumped parameter model criteria. Figure 3 shows agreement between these profiles, which validates the presently obtained exact analytical solution.

Summary and Conclusions

A closed form exact analytical solution of the lumped parameter model of a non-linear heat transfer process, portraying the transient cooling of a spherical body by the combined mechanism of convection and radiation, has been obtained in an implicit form. The obtained exact analytical solution not only shows excellent harmony with its numerical counterparts but also successfully imitates the results of a recently conducted experimental study that depicts the cooling of a metal ball bearing by the combined mechanism of convection and radiation. Use of the lumped parameter model for this experimental study has been found to be valid. While simulating the experimental case study, it has been observed that the convective heat transfer coefficient, possessing temperature-dependent nonlinearity, can easily be replaced by its average value over the whole of the concerned temperature domain without affecting the results in any significant way.

Moreover, the model equation of a simplified case of this problem, recently investigated by various researchers in an approximate manner, has also been exactly solved. Here also, the exact analytical solution shows commendable agreement with the numerical results.

These exact solutions offer a better understanding of the physical process and are found to be valid for all parameter ranges. In addition, these can be very useful in validating the approximate solutions.

Nomenclature

a, b	real and imaginary parts of the complex roots r_3 and r_4
A_1, B_1, C_1, D_1	constants appearing in the partial fraction decomposition
Bi_c	Biot number for convection ($= h_c R/k$)
Bi_r	Biot number for radiation ($= h_r R/k$)
Bi_T	total Biot number for convection and radiation ($= \frac{(h_c + h_r)R}{k}$)
c_p	specific heat of the body, $J/kg \cdot K$
C_1	constant of integration
D	diameter of the ball bearing ($= 2R$), m
h_c	convective heat transfer coefficient, $J/s \cdot m^2 \cdot K$

h_r	radiative heat transfer coefficient $\left(= \frac{\sigma \epsilon (T^4 - T_a^4)}{T - T_a} \right)$, $\text{J/s} \cdot \text{m}^2 \cdot \text{K}$
h_T	combined convective-radiative heat transfer coefficient ($= h_c + h_r$), $\text{J/s} \cdot \text{m}^2 \cdot \text{K}$
k	thermal conductivity of the body, $\text{W/m} \cdot \text{K}$
N_{rc}	dimensionless conduction-radiation parameter, $\sigma \in RT_i^3/k$
r	radial coordinate, m
r_i	i th root
R	radius of the spherical body, m
t	time, s
T	temperature, K
T_a	adiabatic surface temperature, K
T_i	initial temperature of the body, K

Greek Letters

α	thermal diffusivity ($= k/\rho c_p$), m^2/s
ϵ	dimensionless parameter ($= N_{rc}/Bi_c$)
\in	emissivity of ball bearing
η	dimensionless radial coordinate ($= r/R$)
θ	dimensionless temperature ($= T/T_i$)
θ_a	dimensionless adiabatic surface temperature ($= T_a/T_i$)
ρ	density of the spherical body, kg/m^3
σ	Stefan-Boltzmann constant ($= 5.669 \times 10^{-8}$), $\text{W/m}^2 \cdot \text{K}^4$
τ	dimensionless time ($= \alpha t/R^2$)

Subscripts

av	average
f	surrounding fluid
i	initial
max	maximum
s	sink

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Appendix I

Roots of the quartic equation $\frac{Bi_c}{N_{rc}}(\theta_{av} - \theta_a) + (\theta_{av}^4 - \theta_a^4) = 0$ are given as follows:

$$r_1 = \theta_a$$

$$r_2 = \frac{1}{3} \left(-\theta_a - \frac{16^{1/3} \varepsilon \theta_a^2}{\gamma^{1/3}} + \frac{\gamma^{1/3}}{2^{1/3} \varepsilon} \right)$$

$$r_3 = \frac{1}{3} \left(-\theta_a + \frac{2^{1/3} (1 + i\sqrt{3}) \varepsilon \theta_a^2}{\gamma^{1/3}} - \frac{(1 - i\sqrt{3}) \gamma^{1/3}}{16^{1/3} \varepsilon} \right)$$

$$r_4 = \frac{1}{3} \left(-\theta_a + \frac{2^{1/3} (1 - i\sqrt{3}) \varepsilon \theta_a^2}{\gamma^{1/3}} - \frac{(1 + i\sqrt{3}) \gamma^{1/3}}{16^{1/3} \varepsilon} \right)$$

where $\gamma = \left(-27\varepsilon^2 - 20\varepsilon^3\theta_a^3 + 3\sqrt{3}\sqrt{27\varepsilon^4 + 40\varepsilon^5\theta_a^3 + 16\varepsilon^6\theta_a^6} \right)$ and $\varepsilon = \frac{N_{rc}}{Bi_c}$. Following points can be noted regarding the properties of the roots:

- i. The first root is known a priori and has a real positive value, i.e., $r_1 = \theta_a (> 0)$.
- ii. No root is repeated (multiplicity of all the roots is one).
- iii. The second root is negative and real (say, r_2), whereas the third and fourth roots (r_3 and r_4) are complex conjugates.

Appendix II

$$A_1 = \frac{1}{(r_1 - r_2)((r_1^2 - (r_3 + r_4)r_1 + r_3r_4))} = \frac{1}{(r_1 - r_2)((r_1 - a)^2 + b^2)}$$

$$B_1 = \frac{1}{(r_2 - r_1)((r_2^2 - (r_3 + r_4)r_2 + r_3r_4))} = \frac{1}{(r_2 - r_1)((r_2 - a)^2 + b^2)}$$

$$C_1 = \frac{r_1 + r_2 - r_3 - r_4}{(r_3 - r_1)(r_3 - r_2)(r_1 - r_4)(r_2 - r_4)} = \frac{r_1 + r_2 - 2a}{((r_1 - a)^2 + b^2)((r_2 - a)^2 + b^2)}$$

$$D_1 = \frac{r_1r_2 - r_1r_3 - r_2r_3 + r_3^2 - r_1r_4 - r_2r_4 + r_3r_4 + r_4^2}{(r_3 - r_1)(r_3 - r_2)(r_1 - r_4)(r_2 - r_4)} = \frac{r_1r_2 - 2(r_1 + r_2)a + 3a^2 - b^2}{((r_1 - a)^2 + b^2)((r_2 - a)^2 + b^2)}$$