CHANGE OF BASIS VS CHANGE OF COORDINATES

- Let $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$ be some basis in the vector space $V = \mathbb{R}^2$. Then then every vector $\mathbf{b}$ can be uniquely written as a linear combination $\mathbf{b} = y_1\mathbf{u} + y_2\mathbf{v}$.
  The numbers y_1, y_2 are called the *$\mathcal{B}$-coordinates of* $\mathbf{b}$.
- Let $\mathcal{C} = \{\mathbf{i}, \mathbf{j}\}$ be another basis in$V$. For example, let's take $\mathbf{i} = 2\mathbf{u} - 1\mathbf{v}$ and $\mathbf{j} = 1\mathbf{u} + 1\mathbf{v}$.
  This relation between the $\mathcal{B}$- and the $\mathcal{C}$-bases can also be written in matrix form as
  $$\begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$
- If $\mathbf{b} = x_1\mathbf{i} + x_2\mathbf{j}$ is any vector in V, then
  $$\begin{aligned} \mathbf{b} &= x_1(2\mathbf{u} - \mathbf{v}) + x_2(\mathbf{u} + \mathbf{v}) \\ &= (2x_1 + x_2)\,\mathbf{u} + (-x_1 + x_2)\,\mathbf{v}. \end{aligned}$$
  Thus, the transformation from the $\mathcal{C}$-coordinates to the $\mathcal{B}$ coordinates is given by
  $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
- Vector $\mathbf{u}$ spans a line L that consists of all vectors $\mathbf{b} = y_1\mathbf{u} + y_2\mathbf{v}$ whose y_2-coordinate is 0. In terms of the $\mathcal{C}$-coordinates, L consists of all vectors $\mathbf{b} = x_1\mathbf{i} + x_2\mathbf{j}$ such that
  $$y_2 = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$
- Similarly, Span $\{\mathbf{v}\}$ is a line that consists of all vectors $\mathbf{b} = x_1\mathbf{i} + x_2\mathbf{j}$ such that
  $$y_1 = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$
  More generally, any line that is parallel to $\mathbf{u}$ or $\mathbf{v}$ is given by an equation $\mathbf{r}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = const$, where $\mathbf{r}$ is $\begin{bmatrix} -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 2 & 1 \end{bmatrix}$ respectively.

Summary.

- The rows of the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ describe the transformation from the $\mathcal{C}$-coordinates to the $\mathcal{B}$ coordinates:
  $$\begin{aligned} y_1 &= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ y_2 &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$
- The columns of A describe the change from the $\mathcal{B}$-basis to the $\mathcal{C}$-basis:
  $$\mathbf{i} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
- The (red) coordinate grid, that is formed by vectors parallel to $\mathbf{u}$ or $\mathbf{v}$, consists of lines whose $\mathcal{C}$-coordinates satisfy either to the equation
  $$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \text{Const}$$
  or to the equation
  $$\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \text{Const}.$$