## What is the volume of an egg ?

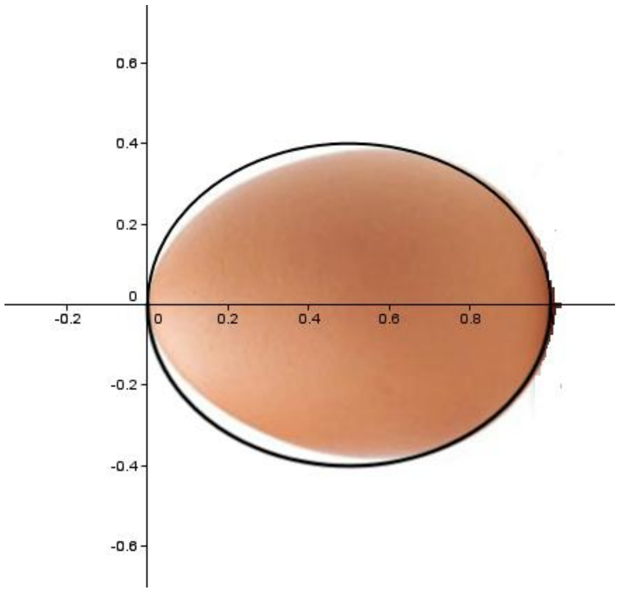
How would you model an egg with an equation ?

Look at the egg shown below. What do you observe ? The outline of this egg does look like an ellipse.

But look carefully, one end of the egg is "bigger" then the other end and thus it has only the horizontal axis of symmetry. Indeed, the shapes of most eggs are not elliptical but they are very close to ellipses.

Given an egg, we can scale the egg in such a way that the longer side of the egg becomes 1 unit long.

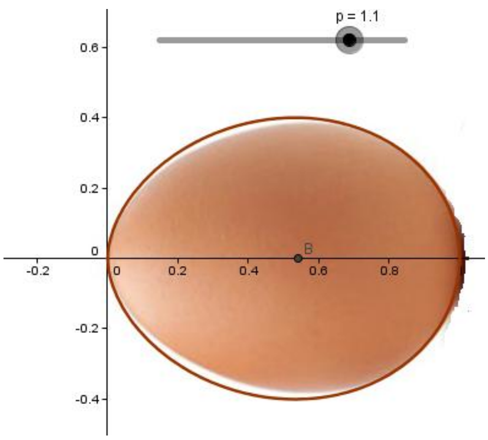
Using Geogebra construction, the equation $\frac{y^2}{0.8^2}+(x-\frac{1}{2})^2=\frac{1}{4}$ fits the "bigger" end of the egg very well.

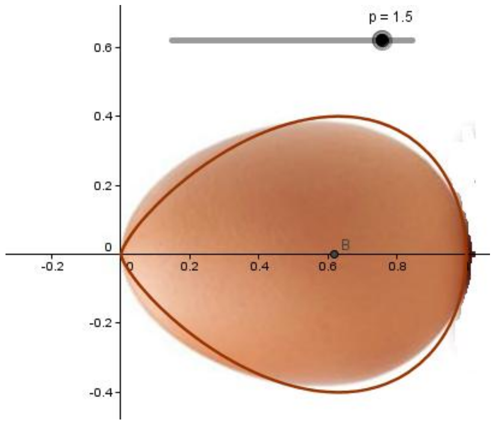



$$\frac{y^2}{0.8^2}+(x-\tfrac{1}{2})^2=\frac{1}{4}$$

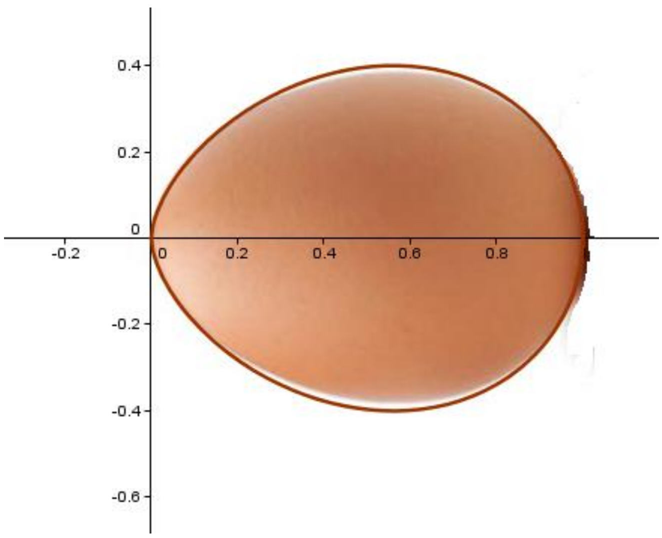
Indeed, without loss of generality, if we let the major axis of an egg to be 1 unit and the minor axis be b units where $0 < b < 1$, an egg can be approximately modelled by the equation $\frac{y^2}{b^2}+(x-\frac{1}{2})^2=\frac{1}{4}$ which is an ellipse with centre at $(\frac{1}{2},0)$ and vertices at $(\frac{1}{2},b)$, $(\frac{1}{2},-b)$, (0, 1) and (0, 0).

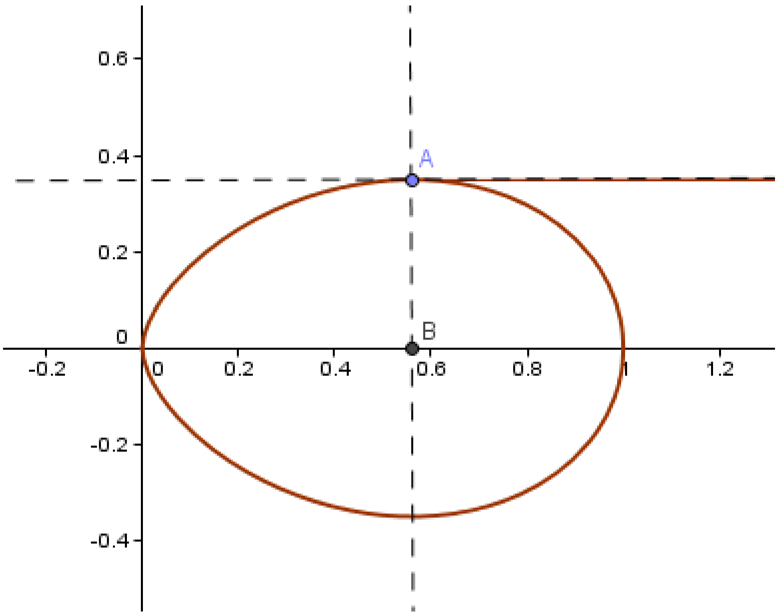
In order to maintain the major axis of the egg as 1 unit and the centre at $(\frac{1}{2},0)$, we transform the variable x to $x^p$ ( that is replacing x by $x^p$ ) where $1 < p < 2$ in the equation $\frac{y^2}{b^2}+(x-\frac{1}{2})^2=\frac{1}{4}$. The diagrams below show the graphs for two different values of p.




Indeed, when p = 1.2, the graph of $\frac{y^2}{b^2}+(x^{1.2}-\frac{1}{2})^2=\frac{1}{4}$ fits the egg quite well.






In practice, the parameter p can be calculated if we know the x-coordinate of the vertical vertex, A denoted by $x_A$. Once we know the value of $x_A$ (the length from the less-rounded end to the vertical vertex along the horizontal axis), we can find the value of p as follows.

$(x_A)^p = \frac{1}{2} \Rightarrow p\ln(x_A) = -\ln 2 \Rightarrow p = -\frac{\ln 2}{\ln x_A}$ and the value of b is just the "minor axis" = 2AB in the diagram.

Once we know the value of b and p, the volume of the egg can be estimated by

$$V = \pi\int_0^1 y^2 dx = \pi b^2 \int_0^1 \left(\frac{1}{4} - (x^p - \frac{1}{2})^2\right) dx$$

$$= \pi b^2 \int_0^1 \left(x^p - x^{2p}\right) dx = \pi b^2 \left[\frac{x^{p+1}}{p+1} - \frac{x^{2p+1}}{2p+1}\right]_0^1 = \frac{\pi p b^2}{(p+1)(2p+1)}$$