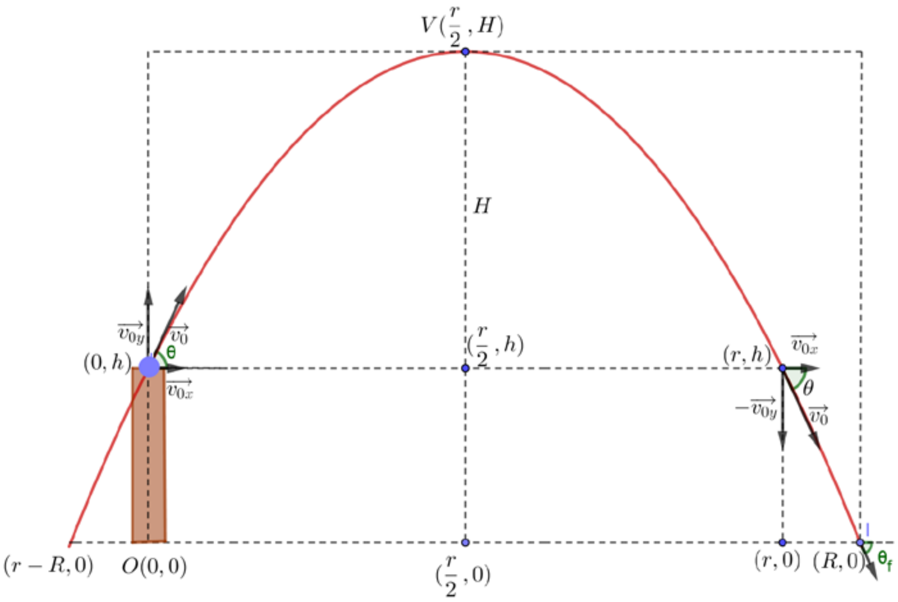
Launching from positive h level and landing at ground level



We use following notations:

R – maximum range (horizontally)

T – total flight time

H – the maximum height of projectile

$(0, h)$ – launching point, level $h > 0$

$(R, 0)$ – landing point, level 0 (ground level)

$V\left(\frac{r}{2}, H\right)$ – vertex of parabola (turning point)

$\left(\frac{r}{2}, 0\right)$ – projection of V on x-axis

We notice that total horizontal movement depends on v_{0x}, h, θ.

Finding range r

Vertical displacement: $y = h + v_{0y}t - \frac{gt^2}{2}$ (uniformly accelerated motion, constant acceleration)

Horizontal displacement: $x = v_{0x}t$ (uniform motion, constant velocity) $\Rightarrow t = \frac{x}{v_{0x}}$.

We replace t and we get $y = h + v_{0y} \cdot \frac{x}{v_{0x}} - \frac{g}{2} \cdot \frac{x}{v_{0x}^2}$

But $v_{0x} = v_0 \cos\theta,\ v_{0y} = v_0 \sin\theta$ and replacing this above we get

$y = h + \operatorname{tg}\theta \cdot x - \frac{g}{2v_0^2 \cos^2\theta} \cdot x^2$.

We notice that line $y = h$ intersect parabola in two points $(0, h)$ and (r, h) with abscises $x = 0$ and $x = r$, so 0 and r are roots of equation $\operatorname{tg}\theta \cdot x - \frac{g}{2v_0^2 \cos^2\theta} \cdot x^2 = 0$. Solving equation we get $x = 0$ and $x = \frac{\mathrm{v}_0^2 \sin 2\theta}{g}$, so $r = \frac{\mathrm{v}_0^2 \sin 2\theta}{g}$.

Finding time t

Movement is done when projectile is landing on ground, so $y = 0$. We replace in vertical displacement equation and we notice that t is positive root of equation $-\frac{gt^2}{2} + v_{0y}t + h = 0$, so total flight time is $T = \frac{v_{0y} + \sqrt{v_{0y}^2 + 2gh}}{g}$ $(*)$.

Let T_r displacement time to the point (r, h). Then $r = T_r \cdot v_{0x}$, or $T_r = \frac{r}{v_{0x}}$.

Replacing r and v_{0x} we get $T_r = \frac{v_0^2 \sin 2\theta}{g v_{0x}} = \frac{v_0^2 \cdot 2\sin\theta\cos\theta}{g v_0 \cos\theta}$, so $T_r = \frac{2v_0 \sin\theta}{g}$.

Finding maximum height H

The projectile reach maximum height H when $t = \frac{T_r}{2}$ (symmetry), $x = \frac{r}{2}$, $y = H$ and $v_y = 0$. But $v_y = v_{0y} - gt$, so $v_{0y} = gt = \frac{gT_r}{2}$. Replacing in vertical displacement formula we get $H = h + gt \cdot t - \frac{gt^2}{2}$, so $H = h + \frac{gt^2}{2}$. But $t = \frac{v_0 \sin\theta}{g}$, hence $H = h + \frac{v_0^2 \sin^2\theta}{2g}$ (1).

From (1) we get $v_0{}^2 \sin^2 \theta = (H-h)2g$ or $v_{0y}{}^2 = 2g(H-h)$, hence $v_{0y} = \sqrt{2g(H-h)}$.

Replacing v_{0y} in $(*)$ we get $T = \dfrac{\sqrt{2g(H-h)} + \sqrt{2g(H-h)+2gh}}{g}$, so $T = \dfrac{\sqrt{2gH} + \sqrt{2g(H-h)}}{g}$ (2).

From horizontal displacement formula $x = v_{0x} \cdot t$ we get $R = v_0 \cos\theta \cdot T$ or $R = \dfrac{v_0 \cos\theta\left(\sqrt{2gH} + \sqrt{2g(H-h)}\right)}{g}$ (3).

Then $v_0 = \dfrac{Rg}{\cos\theta\left(\sqrt{2gH} + \sqrt{2g(H-h)}\right)}$ or $v_0 = \dfrac{R\left(\sqrt{2gH} - \sqrt{2g(H-h)}\right)}{2gh\cos\theta}$ (4)

Let $\overrightarrow{v_i} = \overrightarrow{v_{ix}} + \overrightarrow{v_{iy}}$ velocity in point $(r-R,0)$, and $\overrightarrow{v_f} = \overrightarrow{v_{fx}} + \overrightarrow{v_{fy}}$ velocity in point $(R,0)$. From symmetry $\overrightarrow{v_{ix}} = \overrightarrow{v_{fx}}$ and $\overrightarrow{v_{iy}} = -\overrightarrow{v_{fy}}$. Then $h = 0$ and $y = v_{fy} \cdot t - \dfrac{gt^2}{2}$, and $H = y_{\max} = \dfrac{-\Delta}{4a} = \dfrac{v_{fy}{}^2}{2g}$ hence $v_{fy}{}^2 = 2gH$ (5) or $v_{fy} = \sqrt{2gH}$.

But $v_{fy} = v_f \sin\theta_f$ implies $v_f \sin\theta_f = \sqrt{2gH}$ (6) or $v_f{}^2 \sin^2\theta_f = 2gH$.

Horizontally there is uniform movement, so velocity projection on x-axis is constant $v_{fx} = v_{0x}$ hence $v_f \cos\theta_f = \dfrac{R}{T}$. Replacing in (6) we get $\dfrac{R}{T}\tan\theta_f = \sqrt{2gH}$ hence $\tan\theta_f = \dfrac{T\sqrt{2gH}}{R}$ or $\theta_f = \arctan\left(\dfrac{T\sqrt{2gH}}{R}\right)$ (7).

From (5) and (6) we get $v_f{}^2 = v_{fx}{}^2 + v_{fy}{}^2 = \left(\dfrac{R}{T}\right)^2 + 2gH$, hence $v_f = \sqrt{\left(\dfrac{R}{T}\right)^2 + 2gH}$ (8) or $v_f = \sqrt{v_0{}^2 \cos^2\theta + 2gH}$ $(8')$.