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TANTON'S TAKE ON …

FACTORING TRINOMIALS

JULY 2016

I have to start with a confession. I don't know what a trinomial is.

I can guess that the textbook definition is probably along the lines "an expression that is the sum of three terms, with each term being a number or a variable or a product of numbers and variables." So $a^2+2ab+b^2$ is a trinomial, as is $2x^2+3x+9$ and $x+y+wvyz$ and $14+a^{56}+9(x+1)$ and $\pi+0+1$? (Oh. So x^2 is also a trinomial, since it is equivalent to x^2+0+0?)

I also have to ask: Why do students want to factor them? When is factoring trinomials a burning issue in students' minds? When is it a natural next question in their K-12 mathematical story?

This leads to a second confession: if I had my druthers I would eliminate factoring from the curriculum, most certainly from algebra II at the very, very least.

Now don't get me wrong, I do understand that factoring does come up in odd moments here and there, and that basic factoring such as undoing distribution to see x^2+x as $x(x+1)$ or noticing a difference of two squares, for example, is handy. But a focus on factoring as a curriculum item in and of itself incumbent with a whole slew of special techniques to answer issues that weren't even in one's mind in the first place just seems sad and weird to me. I simply don't accept the rationale: "We do it now because they need to do it later on." (Just do it later on then when it's needed!)

Factoring and Quadratics

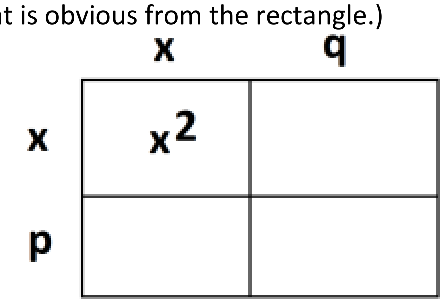
The traditional curriculum insists that factoring be taught in the study of solving quadratic equations: if one can recognize x^2+5x+6 as $(x+2)(x+3)$, then we see that the equation $x^2+5x+6=0$ has solutions $x=-2$ and $x=-3$.

This implies a false construct. The textbook examples here are always expressions with integer coefficients, carefully chosen to always factor into linear terms, and always ones with integer coefficients again. This is extraordinarily artificial. (Does innocent looking x^2-2x-1 factor this way?)

Computer exercise? Let a, b, c each run through the integers one to nine. Of these 729 different quadratic expressions ax^2+bx+c, what percentage of them factor into a pair of linear terms with integer coefficients?

Also, factoring ruins the story of symmetry, which is the compelling message a unit on quadratics should convey. It really does not belong to this story. (See my quadratics notes www.gdaymath.com/courses/ .)

And if, by some chance, the need to recognize x^2+5x+6 as $(x+2)(x+3)$ does happen to come up in some course, simply let students exercise their mathematical might: have them draw a rectangle and nut their way to some numbers that align just beautifully. (No need to mention "two numbers whose product is six that sum to five." Looking for that is obvious from the rectangle.)



Factoring and Algebra II

In factoring a quadratic expression x^2+bx+c with leading coefficient one, one thinks of the factored form $(x+p)(x+q)$. (Why not $(3x+p)\left(\frac{1}{3}x+q\right)$?). So it is natural to associate the region of the rectangle of area x^2 as coming from $x\times x$. It is really quite straightforward to then just follow common sense from the picture in this case. (Try x^2-2x-1 this way?)

But a traditional algebra II course wants students to go further and factor quadratic expressions with leading coefficients different from one as well, that is, to factor expressions of the form ax^2+bx+c or even $ax^2+bxy+cy^2$. (The second one is a quadratic too.)

Question: Why do we need the term "trinomial" if every trinomial discussed in the classroom is a quadratic?

I again have no trouble letting students just draw rectangles and following their noses to find a factored expression. It is rare, very rare, that the need to factor a quadratic actually comes up in a genuine way. So really, playing with a diagram and following common sense really is a sufficiently fruitful approach for those rare occasions.

But traditionalists object to this loose approach of mine. Why? Is it because they, like me, were handed worksheet after factoring worksheet in their day? And, of course, when handed 50 quadratics to factor one wants a procedure or technique to simply get the job done.

But who wants to factor 50 quadratics in the first place? We really do not need to do that to our students ourselves.

TWO TECHNIQUES

Now having made the topic of this essay moot, I will nonetheless say that I am intrigued by the two factoring techniques many teachers teach. I am intrigued as a mathematician. And figuring out why these techniques work makes for a great chapter in an advanced mathematics course for seniors.

So imagine our challenge is to factor ax^2+bx+c, with integer coefficients, into a product of two linear terms, each with integer coefficients.

Are you aware of these two techniques?

SPLIT THE MIDDLE TERM METHOD

Start by identifying two integers p and q that sum to the middle coefficient and whose product equals the product of the first and last coefficients.

For example, to factor $4x^2+16x+15$, we seek integers p and q such that

$$p+q=16$$
$$pq=4\times 15=60.$$

A moment's thought gives $p=10$ and $q=6$. We use these values to now "split the middle term:"

$$4x^2+16x+15=\underline{4x^2+10x}+\underline{6x+15}.$$

Something magical always then happens: We get two groups of terms with a common factor, and when we pull out their respective common factors, what is left is another common factor.

$$\begin{aligned}4x^2+16x+15&=2x(x+5)+3(x+5)\\&=(2x+3)(x+5).\end{aligned}$$

The result is a factored quadratic!

A USEFUL STUDENT TIP?

Finding the factors of numbers can be tricky. For example, factoring $x^2-4x-252$ requires looking for a pair of factors of -252 that sum to -4.

Here's a calculator tip to help students out.

Have students enter the function $f(x)=\dfrac{-252}{x}$ on their calculators and then hit TABLE. Students will then see which integer inputs x give an integer output y. That is, they will see all the integer factor pairs of -252. All they have to do now is identify a pair that sum to -4. In this case, -18 and 14 do the trick and so $x^2-4x-252=(x-18)(x+14)$.

(Or if I gave an example with leading coefficient different from one, our next steps would be to follow the "split the middle term" approach.)

COMMENT: This tip is absurd!

We're saying to students that technology is great for finding factors of numbers. But let's dare not use technology to factor trinomials themselves. It is morally wrong to type $x^2-4x-252$ straight into *Wolfram Alpha*, for example.

A Surprisingly Hard Challenge:

Why does this technique work?

Suppose a,b,c are integers and p and q are integers with $p+q=b$ and $pq=ac$. Can you prove that in writing ax^2+bx+c as $\underline{ax^2+px}+\underline{qx+c}$ and following the procedure demonstrated above we are sure to get a factorization of the quadratic WITH INTEGER COEFFICIENTS? (If you write out the procedure abstractly, it looks like we should be getting fractions left, right, and center!)

Warning: People often miss the subtlety of explaining the first technique. They notice that if you start with the answer $(rx+s)(tx+u)$, a factored expression with integers terms, and expand it you get $rtx^2+(ru+st)x+su$, an expression with middle term split as a sum of two, ru and st, whose product, $rstu$, matches the product of the first and last coefficients.

We want to prove the converse of this: Don't start with the answer, but start instead with $ax^2+px+qx+c$, all coefficients integers, with $pq=ac$. Must you get a factorization that stays with integers? (This really is hard!)

TIC TAC TOE METHOD

Here's a method that removes the grouping and factoring part of the previous method.

To factor $4x^2+16x+15$, say:

1. Draw a tic tac toe board and write the leading coefficient and the constant term in the positions shown.

4	15	

2. Compute their product to complete the row.

4	15	60

3. Think of a pair of factors of the number you just wrote that sum to the middle coefficient of the trinomial. Write them, in some order, to complete the third column of the board.

4	15	60
		10
		6

4. Now we think of four integers to complete the table that multiply across correctly and multiply vertically correctly as shown.

4	15	60
		10
		6

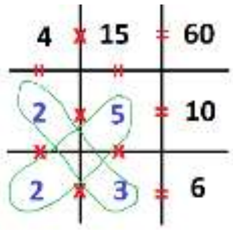
In this example, the six has pairs of factors 1 and 6, and 2 and 3. I am going to choose the latter pair as they also give factors of the 4 and 15, respectively, in the top.

4	15	60
		10
2	3	6

Now I see how to fill in the rest of the table and stay within the world of integers.

4	15	60
2	5	10
2	3	6

5. Now read off the factorization as a cross diagram.

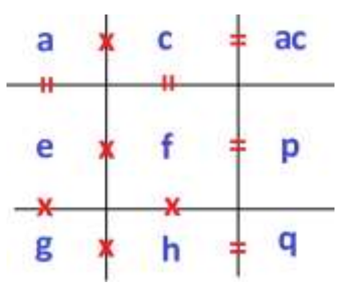


We have

$$4x^2+16x+15=(2x+3)(2x+5).$$

Challenge:
Why does this tic-tac-toe method work? Are there always sure to be integers with which to fill in the table? What if you don't stay with integers, what happens? What happens if we try to apply this technique to x^2-2x-1 which does not factor into linear terms with integer coefficients?

Again, it is easy to start with the end result and see that the final set-up will indeed give a factorization. In the table below we are factoring ax^2+bx+c, and p and q are integers whose product is ac and whose sum is b. The remaining terms are integers.



Then

$$\begin{aligned}(ex+h)(gx+f) &= egx^2+(gh+ef)x+hf\\ &= ax^2+(p+q)x+c\\ &= ax^2+bx+c.\end{aligned}$$

The challenge is in proving that there are sure to be integers that complete the table. (Again, this is hard!)

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Asking why algorithms work often serves as a lovely portal to deep and astounding mathematics. We really don't need computational techniques in this day and age – we have smart phones and internet access. But we do need opportunities to think, and puzzle, and resolve, and push, and extend. Let's use the algorithms we teach for that end. (And in this light, even factoring algorithms become fascinating.)

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