FUNCTIONS

k11707598 fotosearch.com

Equations
X = 4 + 8
Inequalities
1 - 1 > 4 - 6
Function
y = 3x - 2

INPUT x

FUNCTION f:

OUTPUT f(x)
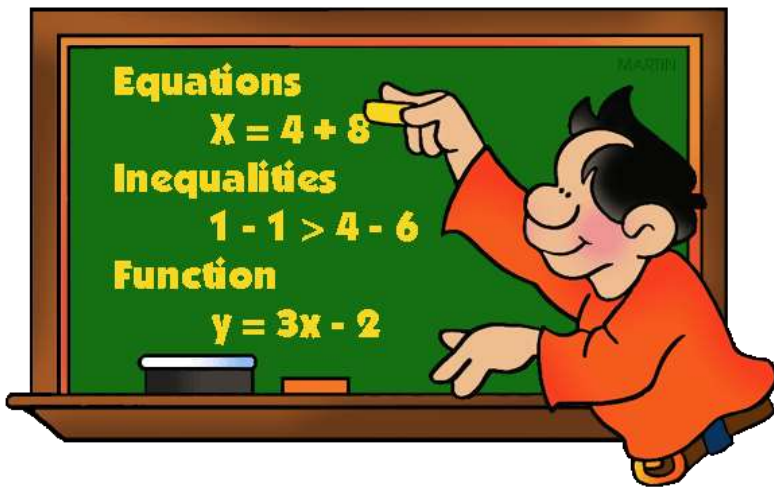
NAME: MR. WAIN
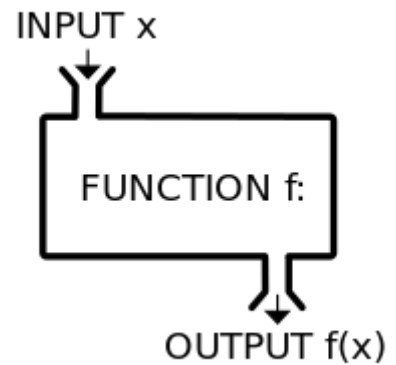
Revision of Solving Polynomial Equations

(i)   "one term in x"

Examples
Solve:

(a)
$$2 - 7x = 12$$
$$-7x = 12 - 2$$
$$-7x = 10$$
$$x = -\frac{10}{7}$$

(b)
$$4x^3 = 16$$
$$x^3 = 4$$
$$x = \sqrt[3]{4}$$
$$x \approx 1.59$$

(c)
$$\frac{17x}{3} - 5 = 12$$
$$\frac{17x}{3} = 17$$
$$17x = 51$$
$$x = 3$$

(ii)   "more than one term in x"

Method:
1) Get the right hand side to equal zero ($= 0$)
2) Eliminate all denominators (where necessary)
3) Factorise Left Hand Side
4) Use Null Factor Law (NFL)

Factorising

- **Quadratic ($ax^2 + bx + c$)**

  - If $b^2 - 4ac$ **(the discriminant, Δ) is:**
    - **Perfect square, use "criss-cross"**
    - **Not a perfect square, but positive use the quadratic formula,**
      $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
    - **Negative, NO REAL solution**

    $\Delta < 0$, no solutions
    $\Delta = 0$, 1 solution
    $\Delta > 0$, 2 solutions

- **Cubic ($ax^3 + bx^2 + cx + d$)**
  - **Use grouping (if possible)**
  - **Factor theorem and long division**

- **Quartic ($ax^4 + bx^3 + cx^2 + dx + e$)**
  - **Use substitution, eg let $a = x^2$**
  - **Factor theorem**

Examples:

Solve the following for x:

1)
$$3x^2 - 7x + 2 = 0$$

$$
\begin{array}{ccc}
3x & & 2 \quad 1 \\
x & & 1 \quad 2
\end{array}
$$

$$(3x - 1)(x - 2) = 0$$

$$3x - 1 = 0 \quad or \quad x - 2 = 0$$

$$3x = 1 \qquad\qquad x = 2$$

$$x = \frac{1}{3}$$

2)
$$x^3 + 4x^2 = 6 - x$$

$$x^3 + 4x^2 + x - 6 = 0$$

$$Let \quad P(x) = x^3 + 4x^2 + x - 6$$

$$P(1) = (1)^3 + 4(1)^2 + (1) - 6 = 0$$

$$\therefore (x - 1) \text{ is a factor of } P(x)$$

$$Long\ division$$

$$
\begin{array}{r}
x^2 + 5x + 6 \\
x - 1 \overline{)\, x^3 + 4x^2 + x - 6} \\
-(x^3 - x^2) \\
\hline
5x^2 + x \\
-(5x^2 - 5x) \\
\hline
6x + 6 \\
-(6x + 6) \\
\hline
0
\end{array}
$$

or				
1	$)\,1$	4	1	-6
$\times$	$\downarrow$	1	5	6
$+$	1	5	6	0

$$(x - 1)(x^2 + 5x + 6) = 0$$

$$(x - 1)(x + 2)(x + 3) = 0$$

$$x = -3, -2, 1$$

3)
$$(3x + 2)^2 + 12(3x + 2) - 13 = 0$$

$$Let\ a = 3x + 2$$

$$a^2 + 12a - 13 = 0$$

$$(a + 13)(a - 1) = 0$$

$$a = -13\ or\ a = 1$$

$$\Rightarrow 3x + 2 = -13\ or\ 3x + 2 = 1$$

$$3x = -15 \qquad\qquad 3x = -1$$

$$x = -5 \qquad\qquad x = \frac{-1}{3}$$

4)
$$x^4 - 2x^2 + 1 = 0$$
$$\text{Let } a = x^2$$
$$a^2 - 2a + 1 = 0$$
$$(a - 1)^2 = 0$$
$$a - 1 = 0$$
$$a = 1$$
$$x^2 = 1$$
$$x = \pm\sqrt{1}$$
$$x = \pm 1$$

5)
$$x^3 + x^2 - 4x - 4 = 0$$
$$x^2(x + 1) - 4(x + 1) = 0$$
$$(x + 1)(x^2 - 4) = 0$$
$$x^2 - 4 = 0 \quad or \quad x + 1 = 0$$
$$x^2 = 4 \qquad\qquad x = -1$$
$$x = \pm 2$$

iii)   Iteration

Use iteration to find the solutions to correct to 1 decimal point.

$Let : P(x) = x^2 - x - 5 \qquad P(0) = -5 \quad P(1) = -5 \quad P(2) = -3 \quad P(3) = 1 \quad P(-1) = -3 \quad P(-2) = 1$

$\qquad\qquad therefore\ solutions\ close\ to\ 3\ \&\ -2$

$try \quad P(2.7) = -0.41 \quad P(2.8) = 0.04 \quad \Rightarrow \quad x = 2.8 \quad is \quad a \quad solution \quad to\ 1\ d.p.$

$try \quad P(-1.7) = -0.41 \quad P(-1.8) = 0.04 \quad \Rightarrow \quad x = -1.8 \quad is \quad a \quad solution \quad to\ 1\ d.p.$

- **Questions:** 22 question sheet. (see Below)
- **Check using** *solve* **command on Ti-NSpire**

- "one term in x"
- "> 1 term in x"

Solve the following:

Question		Hint
1.	$5 - 3x = 16$	-
2.	$1 - \dfrac{1}{x^2} = 0$	*Eliminate the denominator*
3.	$3x^5 = 96$	-
4.	$7x^5 = 200 + 6x^5$	*Collect like terms, answer to 4 d.p*
5.	$\dfrac{2x}{3} - \dfrac{3}{4} = \dfrac{5}{6}$	*Eliminate denominators in one go*
6.	$x - 5 + \dfrac{4}{x} = 0$	-
7.	$2x^2 = 8x + 64$	-
8.	$x^2 + 8x + 8 = 0$	*(i) exact answer* *(ii) correct to 3 d.p.*
9.	$-2x^2 - 6x = 1$	*Correct to 3 d.p.*
10.	$5x^2 + 2x + 1 = 0$	-
11.	$x^3 + 4x^2 + x = 6$	*Make RHS=0, then factor theorem*
12.	$4x^3 + 12x^2 - x - 3 = 0$	*Grouping is quicker*
13.	$x^4 + x^2 - 2 = 0$	*Let a = x^2*
14.	$x^4 - 14x^2 = -1$	*Make RHS=0, let a =x^2, ans. to 3 d.p.*
15.	$x^4 + x^3 - 7x^2 - x + 6 = 0$	*Grouping two and three*
16.	$(3x + 1)^2 = 3x + 1$	*Let a =3x+1*
17.	$x^2 + 1 + \dfrac{2}{x} = 0$	*Eliminate denominator then factor theorem*
18.	$x^2 = \dfrac{1}{1 + 2x^2}$	*Eliminate denominator then let a=x^2*
19.	$6(x^3 - x^5) = 0$	*Eliminate constant, HCF*
20.	$\dfrac{1}{3}(x + 1)^4(5 - 2x) + 2 = 2$	*Eliminate constants*

For the following use iteration to find solutions correct to 1 decimal place (1 d.p)

21.	$x^2 + x - 7 = 0$	
22.	$x^3 + x - 5 = 0$	

Completing The Square

Completing the square allows a quadratic of the form $y = ax^2 + mx + n$, to be written in the Turning Point form, $y = a(x-b)^2 + c$.

Examples: Express the following in the Turning Point form $y = a(x-b)^2 + c$.

1)
$$y = x^2 + 4x + 1$$
$$y = x^2 + 4x + 4 - 4 + 1$$
$$y = (x+2)^2 - 3$$

2)
$$y = x^2 - 3x + 1$$
$$y = x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 1$$
$$y = (x - \frac{3}{2})^2 - \frac{5}{4}$$

3)
$$y = -3x^2 - 12x - 7$$
$$y = -3\left(x^2 + 4x + \frac{7}{3} \right)$$
$$y = -3\left(x^2 + 4x + 4 - 4 + \frac{7}{3} \right)$$
$$y = -3\left[(x+2)^2 - \frac{5}{3} \right]$$
$$y = -3(x+2)^2 + 5$$

- **Ex 4A Q2, 4 (Only Complete the Square)**
- **CAS Calculator:** *expand* **command to check answers & the** *Complete The Square* **command**

Factor Theorem

Examples

1) Without performing long division, find the remainder when $2x^3 - 8x + 11$ is divided by $(x+3)$.

$$Let \quad P(x) = 2x^3 - 8x + 11$$
$$P(-3) = 2(-3)^3 - 8(-3) + 11$$
$$= -54 + 24 + 11$$
$$= -19$$

So the remainder is -19.

2) Find "a", given that when $x^3 - 2x + a$ is divided by $(x-2)$ the remainder is 7.

$$Let \quad P(x) = x^3 - 2x + a$$
$$P(2) = 7$$
$$\therefore 7 = (2)^3 - 2(2) + a$$
$$7 = 8 - 4 + a$$
$$3 = a$$

Rational Root Theorem

- Sometimes there are no integer solutions to a polynomial, but there maybe rational solutions.
- e.g. if $P(x) = 2x^3 - x^2 - x - 3$, we can show $P(1) \neq 0$, $P(-1) \neq 0$, $P(3) \neq 0$, $P(-3) \neq 0$.
- So there is no integer solution.
- So next we try $P\left(\dfrac{3}{2}\right)$, $P\left(-\dfrac{3}{2}\right)$, $P\left(\dfrac{1}{2}\right)$ & $P\left(-\dfrac{1}{2}\right)$ and will discover $P\left(\dfrac{3}{2}\right) = 0$ therefore $(2x - 3)$ is a factor of $P(x)$.
- Unsure of signs then solve the equation $2x - 3 = 0$ for .

Example: Use the Rational-root theorem to help factorise $P(x) = 3x^3 + 8x^2 + 2x - 5$

$$P(x) = 3x^3 + 8x^2 + 2x - 5 \qquad P(1) = 8 \qquad P(-1) = -2 \qquad P(5) = 580 \qquad P(-5) = -190 \qquad P\left(-\dfrac{5}{3}\right) = 0$$

$$So \quad (3x+5) \text{ is a factor.} \qquad \dfrac{3x^3 + 8x^2 + 2x - 5}{3x + 5} = x^2 + x - 1 \implies 3x^3 + 8x^2 + 2x - 5 = (3x+5)(x^2 + x - 1)$$

$$x^2 + x - 1 = \left(x + \dfrac{1}{2}\right)^2 - \dfrac{5}{4} \text{ by completing the square} \qquad \left(x + \dfrac{1}{2}\right)^2 - \dfrac{5}{4} = \left(x + \dfrac{1}{2} + \dfrac{\sqrt{5}}{2}\right)\left(x + \dfrac{1}{2} - \dfrac{\sqrt{5}}{2}\right)$$

$$\implies 3x^3 + 8x^2 + 2x - 5 = (3x+5)(x^2 + x - 1) = (3x+5)\left(x + \dfrac{1}{2} + \dfrac{\sqrt{5}}{2}\right)\left(x + \dfrac{1}{2} - \dfrac{\sqrt{5}}{2}\right)$$

$$Note : \text{The solutions would be} \quad -\dfrac{5}{3}, -\dfrac{1}{2} \pm \dfrac{\sqrt{5}}{2} \quad or \quad -\dfrac{5}{3}, \dfrac{-1 \pm \sqrt{5}}{2}$$

- **Ex4D Q 2, 4, 7, 8, 10, 11, 16, 17, 20**

Straight Lines/Simultaneous Equations

- The gradient of a straight line is always constant.

- Gradient $m = \dfrac{rise}{run} = \dfrac{y_2 - y_1}{x_2 - x_1}$

- Distance between 2 points: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Pythagoras)

"m" is negative   "m" is positive   "m" is zero   "m" is undefined
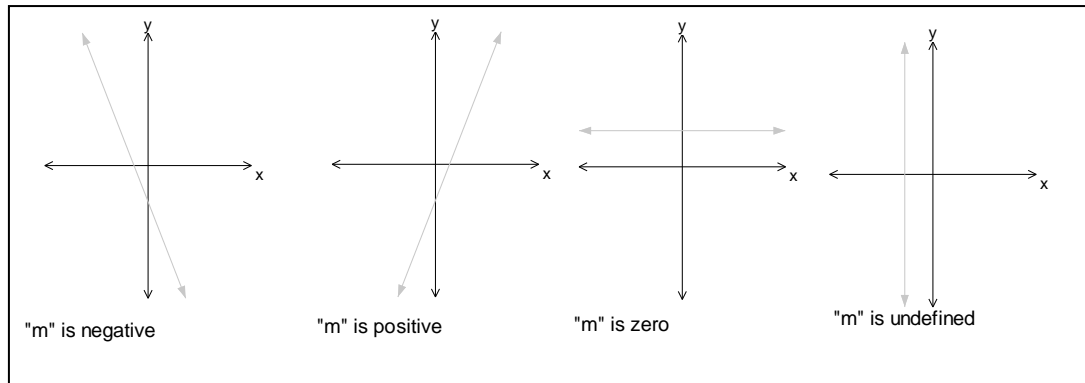
- Midpoint, M, of two points is given by $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right)$

- If m_1 is the gradient of a straight line and m_2 is the gradient of another straight line…
   - If the two lines are parallel then $m_1 = m_2$
   - If the two lines are perpendicular then $m_1 \times m_2 = -1$ or $m_1 = \dfrac{-1}{m_2}$

Equation of a straight line:
- To find the equation of a straight line, you need:
   - The gradient (m) and the Y-intercept (c), then use $y = mx + c$
   - The gradient (m) and the coordinates of one point on the line (x_1, y_1), then use
     $y = mx + c$ or $y - y_1 = m(x - x_1)$.
   - The coordinates of two points on the line (x_1, y_1) and (x_2, y_2), then use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, and
     then use $y = mx + c$ or $y - y_1 = m(x - x_1)$.
   -

Example: Find the equation of the line passing through (-2, -3) and (2, 5).

$Let \quad (x_1, y_1) = (-2, -3) \quad and \quad (x_2, y_2) = (2, 5)$

$$m = \frac{5 - (-3)}{2 - (-3)} = \frac{8}{4} = 2$$

$$\Rightarrow y - (-3) = 2(x - (-2))$$
$$y + 3 = 2(x + 2)$$
$$y + 3 = 2x + 4$$
$$y = 2x + 1$$

- **Ex2C Q 1, 2, 3, 5, 10, 11, 12, 13, 14, 16, 23**

Simultaneous Equations

- **3 situations**
    - **No solutions**
    - **Infinitely many solutions**
    - **A unique solution**

1. No solution
    - Means the lines are parallel
    - They have the same gradient <u>but</u> a different Y-intercept
    - e.g. $2x + y = 2$ & $2x + y = 5$
2. Infinitely many solutions
    - Means you have the same line
    - e.g. $2x + y = 2$ & $4x + 2y = 4$
3. A unique solution
    - Means the lines are different and meet at one point only.
    - e.g. $2x + y = 2$ & $x + y = 5$

Example 1: Explain why the following pair of simultaneous equations have no solutions

$2x + 3y = 6$ & $4x + 6y = 24$

$2x + 3y = 6$ $\qquad\qquad\qquad$ $4x + 6y = 24$

$\qquad 3y = -2x + 6$ $\qquad\qquad\qquad$ $6y = -4x + 24$

$\qquad y = \dfrac{-2x}{3} + 2$ $\qquad\qquad\qquad$ $y = \dfrac{-4x}{6} + 4$

$\qquad\qquad\qquad \Rightarrow$ $\qquad\qquad\qquad\qquad$ $y = \dfrac{-2x}{3} + 4$

Parallel lines, same gradient, different Y - Intercept

Example 2:

Consider the system of simultaneous equations given by:

$$mx + y = 2$$
$$2x + (m-1)y = m$$

Find the value(s) of m for which there is no solution.

$mx + y = 2$ — (1)

$2x + (m-1)y = m$ — (2)

for no solutions or infinite solutions the determinant of the matrix $\begin{bmatrix} m & 1 \\ 2 & m-1 \end{bmatrix}$ equals zero

that is $\begin{vmatrix} m & 1 \\ 2 & m-1 \end{vmatrix} = 0$

$m(m-1) - 2 \times 1 = 0$

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = 2 \quad or \quad m = -1$

$Check: m = 2$

(1): $2x + y = 2$

(2): $2x + y = 2$ they are the same equation, hence an infinite number of solutions.

$m = -1$

(1): $-x + y = 2$

(2): $2x - 2y = -1 \implies -x + y = \dfrac{-1}{2} \implies$ parallel lines (same gradient, different y - int)

$\therefore$ no solution (this is the answer required).

Note: For a unique solution the determinant $\neq 0$. For the above example:

$=>$ the values of m for which there is a unique solution, $m \in R \backslash \{-1, 2\}$

$m(m-1) - 2 \neq 0$

...

$(m-2)(m+1) \neq 0$

$m \neq -1, 2$

==Simultaneous Linear Equations Worksheet==

1. Consider the system of simultaneous linear equations given by

   (a) $\begin{aligned} mx + 12y &= 24 \\ 3x + my &= m \end{aligned}$

   (b) $\begin{aligned} (m+1)x + 2y &= 0 \\ 4x + (m-1)y &= m \end{aligned}$

   Find the value(s) of m for which there is a unique solution.

2. Consider the system of simultaneous linear equations given by

   (a) $\begin{aligned} (m-1)x + 5y &= 7 \\ 3x + (m-3)y &= 0.7m \end{aligned}$

   (b) $\begin{aligned} (m-3)x - 5y &= -2 \\ 2x - (2m+2)y &= -4 \end{aligned}$

   Find the value(s) of m for which there are infinitely many solutions.

3. Consider the system of simultaneous linear equations given by

   (a) $\begin{aligned} mx + 2y &= 6 \\ 3x + (m-1)y &= 6 \end{aligned}$

   (b) $\begin{aligned} 5x + (m-3)y &= 1 \\ mx + 2y &= m \end{aligned}$

   Find the value(s) of m for which there is no solution.

Answers:

1 (a) $m \in R \backslash \{-6, 6\}$ (b) $m \in R \backslash \{-3, 3\}$

2 (a) $m = 6$ (b) $m = 4$

3 (a) $m = -2$ (b) $m = -2, 5$

- ==**Ex 2F 3, 4, 5, 6**==

Sets

Notation
- A set is a collection of objects
- The objects are known as elements
  - If x is an element of A, $x \in A$
  - $y \notin A \Rightarrow y$ is not an element of A or
    $2 \notin$ the set of odd numbers
- If something is a subset of A, for example B, $B \subseteq A$. (Boys in the Year 12 Methods class is an example of a subset)
- If 2 sets have common elements, it is called an intersection ($\cap$) ie $A \cap B$.
- $\varnothing$ it the empty set.
- $\cup$, union, $A \cup B$ is the set of elements that are either in A or B.
- The **set difference** of two sets A and B is given by $A \backslash B = \{x: x \in A,\ x \notin B\}$. Means what's in A but not in B.

Example

> If $A = \{1,2,3,7\}$ and $B = \{3,4,5,6,7\}$
>
> (i) Find $(a) A \cap B$ $(b) A \cup B$ $(c) A \backslash B$ $(d) B \backslash A$
>
> $(a)\ A \cap B = \{3,7\}$
>
> $(b)\ A \cup B = \{1,2,3,4,5,6,7\}$
>
> $(c)\ A \backslash B = \{1,2\}$
>
> $(d)\ B \backslash A = \{4,5,6\}$
>
>
> (ii) True or False
>
> (a) $3 \in A$ True
>
> (b) $5 \in A$ False
>
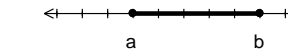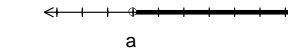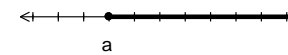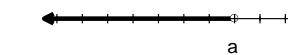> (c) $6 \notin B$ False
>
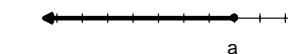> (d) $\{4,5\} \subseteq A$ False
>
> (e) $\{4,5\} \subseteq B$ True
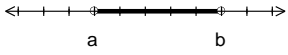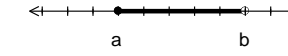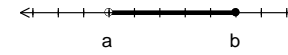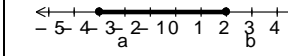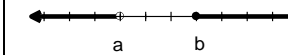
Sets of Numbers
- **N**, the set of Natural Numbers $\{1, 2, 3, 4, \ldots..\}$ is a subset of...
- **Z**, the set of Integers $\{\ldots.,-2, -1, 0, 1, 2, \ldots.\}$ is a subset of…
- **Q**, the set of Rational numbers, numbers which can be expressed in the form $\dfrac{m}{n}$ is a subset of…
- **R**, the set of Real numbers
- $N \subseteq Z \subseteq Q \subseteq R$
- **Q'**, is the set of irrational numbers, eg, $\sqrt{2}, \pi, e$
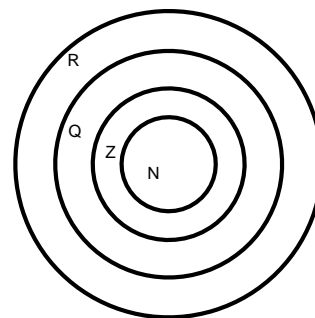
Subsets of the Real numbers

Set	Interval	Number Line
$\{x: a < x < b\}$	(a, b)	a b
$\{x: a \le x < b\}$	[a, b)	a b
$\{x: a < x \le b\}$	(a, b]	a b
$\{x: a \le x \le b\}$	[a, b]	a b
$\{x: x > a\}$	(a, ∞)	a
$\{x: x \ge a\}$	[a, ∞)	a
$\{x: x < a\}$	(-∞, a)	a
$\{x: x \le a\}$	(-∞, a]	a

Example: Complete:

	Set	Interval	Number Line
A	$\{x: x>2\}$		
B		[-2,3]	
C			− 5 − 4 − 3 − 2 − 1 0 1 2 3 4 5 a b
D		(-∞, 5]	
E			a b
F		R^{+}	
G	$\{x: x < 0\}$		
H		$R \setminus \{0\}$	

Relations and Functions

Definition of a function
- Any relation in which <u>no two</u> ordered pairs have the same first element (ie x – value).
    - The x value is only used once
    - {(1,2), (2,4), (3,6), (4,8)} is a function
    - {(-2,0), (-1, -3), (-1, 3), (0,-2), (0,2)} is not a function
- A function is a relation with one-to-one correspondence or many-to-one correspondence.
- Eg of that: $f(x) = 3x^2$ *or* $y = 2x + 4$ *or* $g(x) = \sqrt{x-2}$
- Functions are a subset of relations (one-to-many $y^2 = x$ or many-to-many $x^2 + y^2 = 4$)
- If a relation is represented graphically, apply a "**vertical line**" test to decide whether it is a function or not
    - Cuts the graph once – function
    - Cuts the graph more than once - not a function
- The first elements of the ordered pair in a function makes the set called the *DOMAIN*.
- The second elements make the set called the *RANGE*.
- Some other terms used: Image (y), pre-image (x), $(x, y) \in f$

Notation for description of a Function
- $f : R \rightarrow R, \quad f(x) = 3x^2$
- f is the name of the function (use f, g, h), : means such that
- R is the domain (be careful for restrictions of the domain)
- $\rightarrow R$ is the possible values that the domain can map onto (it is not the actual range)
- $f(x) = 3x^2$ represents the rule

Example: Rewrite the following using the function notation
$\{(x, y) : h(x) = 5x^2 + 7, x > -2\}$

$$\boxed{h : (-2,0) \rightarrow R, h(x) = 5x^2 + 7}$$

Example: For the function with the rule $g(x) = 4x^2 + 5$, evaluate:

(i)	g(2)	(ii)	g(-3)	(iii)	g(0)
(iv)	g(a)	(v)	g(x+h)	(vi)	g(x)=9

$$
\begin{array}{ll}
(i) \quad g(2) = 4(2)^2 + 5 = 16 + 5 = 21 & (ii) \quad g(-3) = 4(-3)^2 + 5 = 36 + 5 = 41 \\
(iii) \, g(0) = 4(0)^2 + 5 = 5 & (iv) \quad g(a) = 4(a)^2 + 5 = 4a^2 + 5 \\
\end{array}
$$

$(v)\ g(x+h) = 4(x+h)^2 + 5 = 4(x^2 + 2xh + h^2) + 5 = 4x^2 + 8xh + 4h^2 + 5$

$(vi)\ 4x^2 + 5 = 9$

$4x^2 = 4$

$x^2 = 1$

$x = \pm 1 \qquad\qquad$ -1 and +1 are the pre-image of 9.

Ex1B Q 1, 2 cef, 3, 4, 5, 6, 7, 8, 9cde, 10, 11, 12abc, 13, 14, 15, 16

One to One Functions

- **VERTICAL LINE TEST** – to see if we have a function
- **HORIZONTAL LINE TEST** – to see if we have a one-to-one function
- **Examples:**
  - Parabola
  - Cubic
  - Exponential

Implied domains

- Often the domain is not stated for a function.
- Assume the domain is to be as large as possible (i.e. select from R)
- **Examples:**
  - $y = 3x^2$, the implied domain is R as all values of x can be used
  - $y = \sqrt{x}$, the implied domain is $[0, \infty)$
  - $y = \sqrt{x-2}$, $[2, \infty)$
  - $y = \sqrt{2-x}$, $(\infty, 2]$
  - $y = \dfrac{1}{x}$, $R \setminus \{0\}$
  - $y = \dfrac{4}{2-x}$, $R \setminus \{2\}$
  - $y = \sqrt{x^2 - 7x + 12}$, $(-\infty, 3] \cup [4, \infty)$ look at graph of parabola – what is under the root.
  - $y = \dfrac{1}{x^2 - 7x + 12}$, $R \setminus \{3,4\}$

- **Ex 1C Q 1, 2, 3, 4, 5, 6, 7, 8**

Hybrid functions (Piecewise functions)

- A function which has <u>different rules</u> for <u>different subsets</u> of the domain.

Example: Sketch the graph of:

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 < x \le 2 \\ x+2, & x > 3 \end{cases}$$ and state the domain and range.
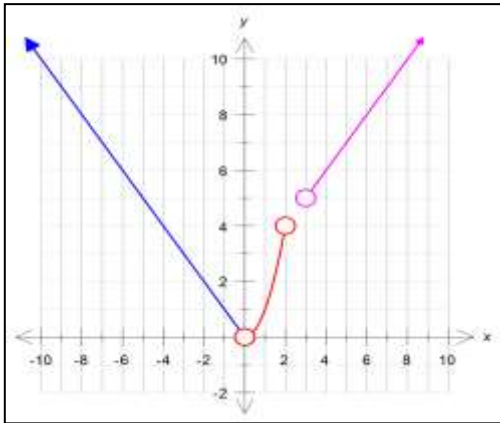
Domain $= (-\infty, 0) \cup (0, 2] \cup (3, \infty)$

Range $= (0, \infty)$ or R^+

For the above function find:
(a) $f(-4)$ (b) $f(1)$ (c) $f(2a)$

(a) $\begin{aligned} f(-4) &= -(-4) \\ &= 4 \end{aligned}$ (b) $\begin{aligned} f(1) &= (1)^2 \\ &= 1 \end{aligned}$

(c)

(i) $-2a, \quad 2a < 0 \Rightarrow a < 0$

(ii) $(2a)^2, \quad 0 < 2a < 2$

$\quad\quad 4a^2, \quad 0 < a < 1$

(iii) $2a + 2, \quad 2a > 3 \Rightarrow a > \dfrac{3}{2}$

$$f(2a) = \begin{cases} -2a, & a < 0 \\ 4a^2, & 0 < a \le 1 \\ 2a+2, & a > \dfrac{3}{2} \end{cases}$$

- *Ti-NSpire* : From the templates, select the hybrid with 3 choices. (see p18 of text).

Odd & Even Functions

- An odd function is defined by:

$$f(-x) = -f(x) \text{ this is the TEST to see if ODD/EVEN}$$
$$if \quad f(x) = x^3 - x$$
$$\Rightarrow f(-x) = (-x)^3 - (-x)$$
$$f(-x) = -x^3 + x = -(x^3 - x) = -f(x) \therefore it \ is \ odd$$
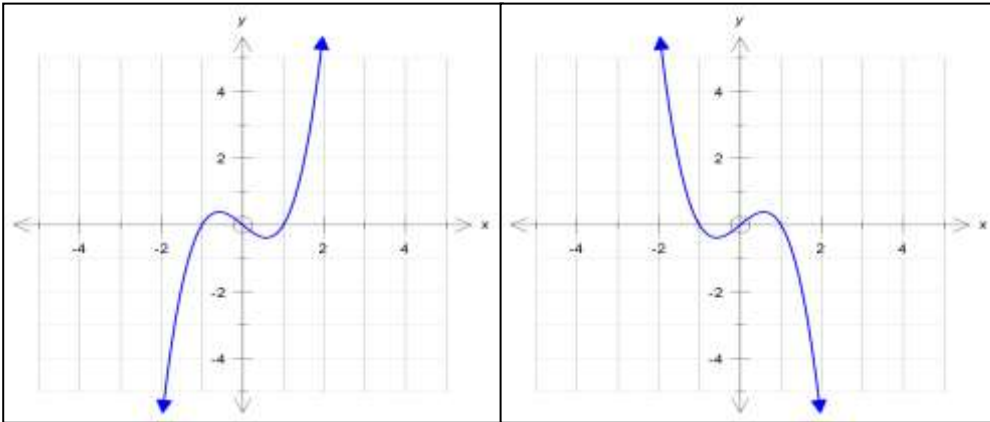
- Can also consider an Odd function has 180° rotational symmetry about the origin.

- An even function is defined by:

$$f(-x) = f(x)$$
$$if \quad f(x) = x^2 - 1$$
$$\Rightarrow f(-x) = (-x)^2 - 1$$
$$f(-x) = x^2 - 1 = f(x) \therefore it \ is \ even$$

- **Ex 1C Q 9 , 10 , 11 , 12, 13, 14, 15, 16, 17, 18, 19**

Sums and Products of Functions

Example: If $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{4-x}$ find:
(a) $(f+g)(x)$
(b) $(f+g)(3)$
(c) $(fg)(x)$
(d) $(fg)(3)$

(a) $(f+g)(x) = f(x) + g(x) = \sqrt{x-2} + \sqrt{4-x}$
Implied domain for $f = [2,\infty)$ and the implied domain for $g = (-\infty,4]$
$\Rightarrow$ only defined domain for $f+g = [2,4]$
(b) $(f+g)(3) = \sqrt{3-2} + \sqrt{4-3} = \sqrt{1} + \sqrt{1} = 2$
(c) $(fg)(x) = f(x) \times g(x) = \sqrt{x-2} \times \sqrt{4-x} = \sqrt{(x-2)(4-x)},\ dom\ fg = [2,4]$
(d) $(fg)(3) = \sqrt{(3-2)(4-3)} = \sqrt{1 \times 1} = 1$

Graphing by Additions of Ordinates

This involves the addition of the y-values of the given equations.

For example, if $f(x) = \sqrt{x}$ and $g(x) = 1 - x$ the graph of $y = f(x) + g(x)$ is obtained by adding the y-values for every value of x for which both curves simultaneously exist.

For $f(x) = \sqrt{x}$ the domain is $[0,\infty)$

For $g(x) = 1 - x$ the domain is $(-\infty,\infty)$

Therefore, the values of x for which both curves are defined simultaneously is given by
$[0,\infty)$

Sketch the two graphs above, on graph paper, see blackboard for specific instructions.
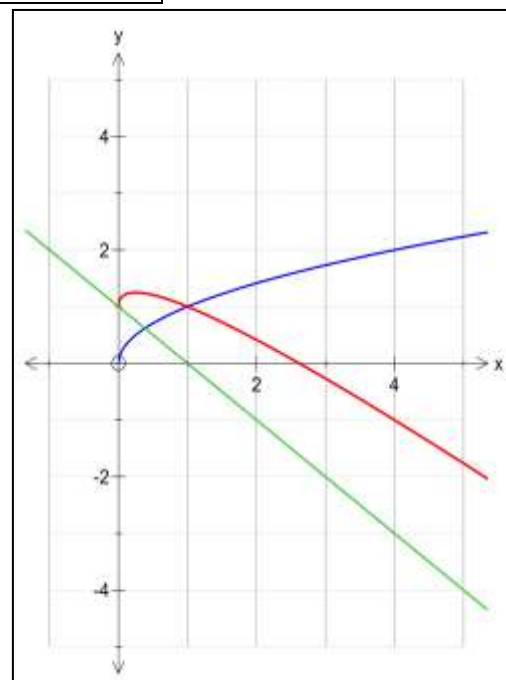
Adding the y-values is straight forward as long as you know the equations of the graphs. However, you need to be able to add two graphs without this information.

Hints: when using the addition of ordinates.

1. Look for regions where both graphs are positive
(ie both lie above the x-axis)
(this means that when you add the y-values, you will obtain a larger positive y-value)

2. Look for regions where both graphs are negative (ie both lie below the x-axis)
(this means that when you add the y-values, you will obtain a more negative y-value)

3. Consider the regions where the graphs differ in sign and then be discerning in where the sum of the two values lie.

4. Look for asymptotic behaviour.

If you are asked to find $f(x) - g(x)$, it is easier to sketch $f(x) + (-g(x))$, that is, reflect $g(x)$ in the x-axis and continue as above.
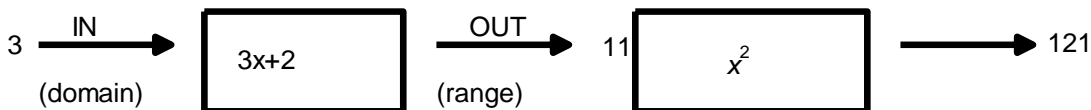
- **Ex1D Q 1, 2, 3, 4, 6, 8a, 10, 12**

Composite functions

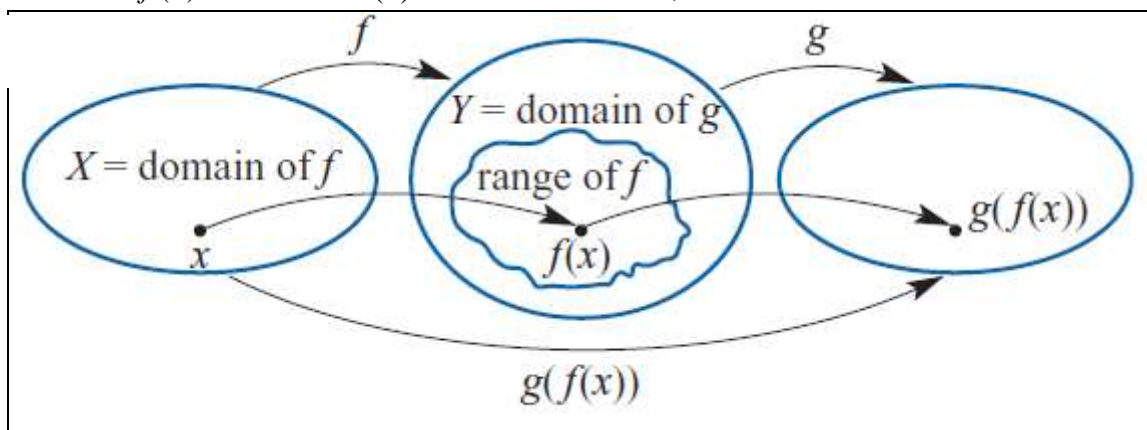- Think of a function machine, eg $f(x) = 3x + 2$ and find $f(3)$.
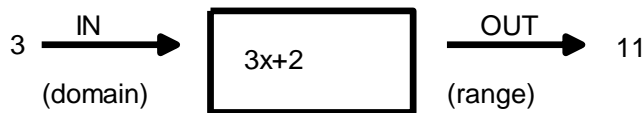
- What happens if we use 2 machines, eg $f(x) = 3x + 2$ and $g(x) = x^2$

- New function has been defined, $h(x)$,

    $h(x) = (3x + 2)^2$

- $h(3) = 121$ or $h(3) = g(f(3)) = g(11)$
- $h(-2) = g(f(-2)) = g(-4) = 16$
- $\Rightarrow h$ is said to be the composition of g with f.
- $h = g \circ f$ or $h(x) = g(f(x))$
- The domain of h or $h(x)$ is the domain of f.
- Consider $f(x) = x - 3$ and $h(x) = \sqrt{x}$. When $x = 4$, $x = 2$

-

Example: Find both $f \circ g$ and $g \circ f$, stating the domain and range of each, if $f : R \to R$, $f(x) = 2x - 1$ and $g : R \to R$, $g(x) = 3x^2$.

	Domain	Range
f	R	R
g	R	$R^+ \cup \{0\}$

$$f \circ g = f(g(x))$$
$$= 2(3x^2) - 1$$
$$= 6x^2 - 1$$
$$\text{dom } f \circ g = R$$
$$\text{ran } f \circ g = [-1, \infty)$$

$$g \circ f = g(f(x))$$
$$= 3(2x - 1)^2$$
$$\text{dom } g \circ f = R$$
$$\text{ran } g \circ f = [0, \infty)$$

$f \circ g$ is defined since $\text{ran } g \subseteq \text{dom } f$ and $g \circ f$ is defined since $\text{ran } f \subseteq \text{dom } g$

Example 2: If $g(x) = 2x - 1$, $x \in R$ and $f(x) = \sqrt{x}$, $x \geq 0$

        (a) state which of $f \circ g$ and $g \circ f$ is defined

        (b) state the domain and rule of the defined.

	Domain	Range
f	$R^+ \cup \{0\}$	$R^+ \cup \{0\}$
g	R	R

(a)
$$\boxed{\begin{array}{l} ran\ g \not\subset dom\ f \\ \Rightarrow f \circ g \text{ is not defined} \\ ran\ f \subseteq dom\ g \\ \Rightarrow g \circ f \text{ is defined} \end{array}}$$

(b)
$$\boxed{\begin{array}{l} g \circ f = 2\sqrt{x} - 1 \\ \Rightarrow dom\ g \circ f = dom\ f = R^+ \cup \{0\} \end{array}}$$

Example 3: for $f(x) = x^2 - 1$ and $g(x) = \sqrt{x}$,

        (a) Is (i) $g \circ f$ defined, (ii) $f \circ g$ defined?

        (b) Determine a restriction for f, f^*, so that $g \circ f^*$ is defined.

	Domain	Range
f	R	$[-1, \infty)$
g	$R^+ \cup \{0\}$	$R^+ \cup \{0\}$
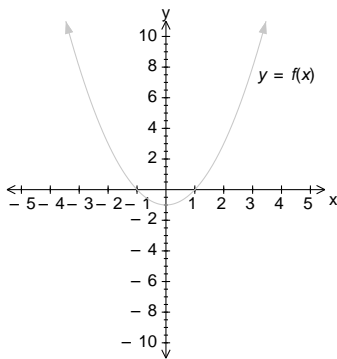
(a)
$$\boxed{\begin{array}{l} ran\ g \subseteq dom\ f \\ \Rightarrow f \circ g \text{ is defined} \\ ran\ f \not\subset dom\ g \\ \Rightarrow g \circ f \text{ is not defined} \end{array}}$$

(b)
$$\boxed{\begin{array}{l} \text{For } g \circ f \text{ to be defined } ran\ f \subseteq dom\ g \\ \Rightarrow dom\ f^* = (-\infty, -1] \cup [1, \infty)\ or\ R \setminus (-1,1) \\ \Rightarrow f^*: R \setminus (-1,1) \to R,\ f^* = x^2 - 1 \\ \Rightarrow g \circ f^* = g(f^*(x)) \\ = g(x^2 - 1) \\ = \sqrt{x^2 - 1} \\ \therefore dom\ g \circ f^* = dom\ f^* = R \setminus (-1,1) \end{array}}$$

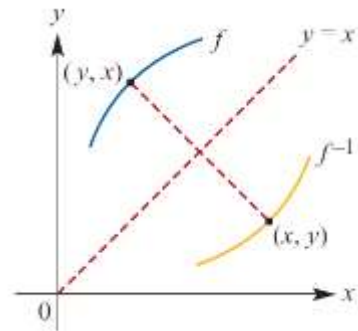- **Ex1E Q 1a - e, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12**
- **CAS Calculator: composite function: define *f(x)* & *g(x)* then *f(g(x))***

Inverse Functions

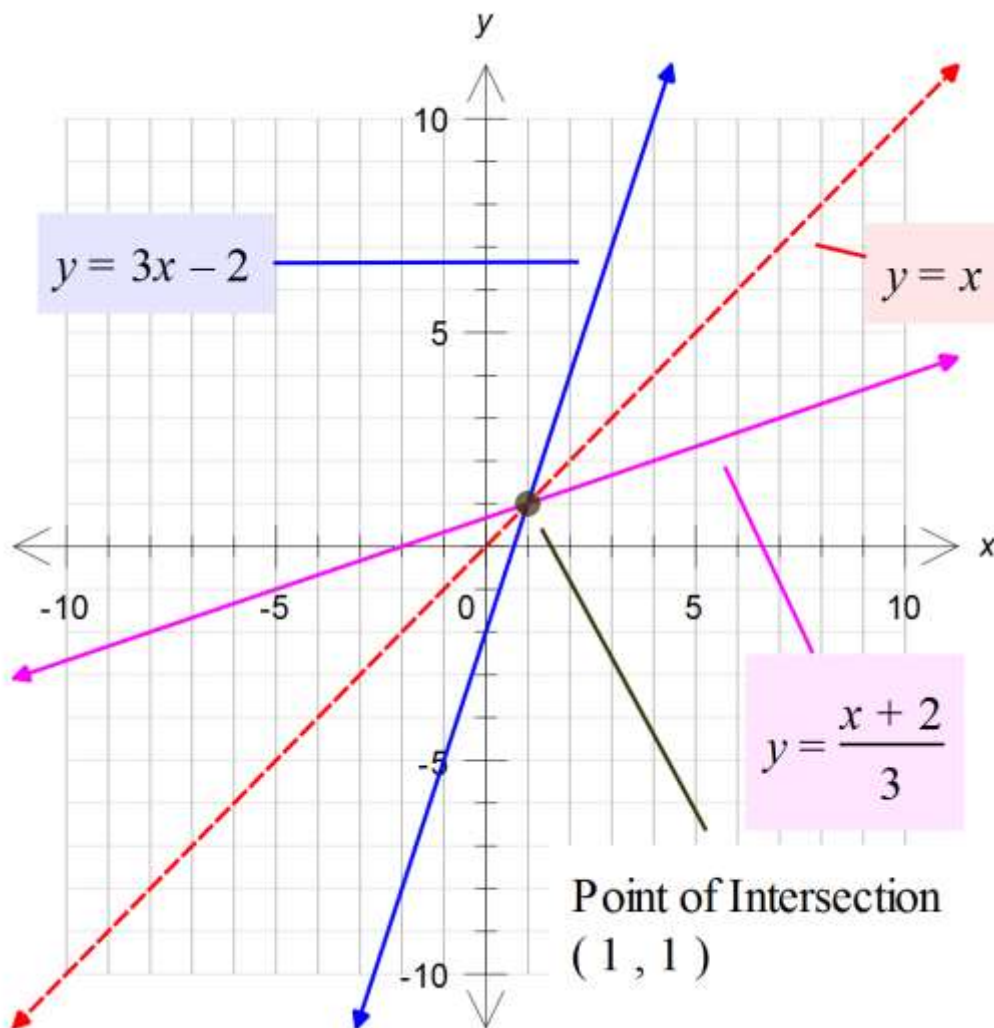The Inverse of a function $\qquad y = f^{-1}(x)$

For the function $y = f(x)$

- The graph of its inverse $y = f^{-1}(x)$ is found by reflecting the original in the line $y = x$.
- The rule of its inverse is found by swapping x for y (and then making y the subject of the equation.

Example: For the function
$f : [-1, 3] \to R$ where $f(x) = 3x - 2$

         a. Sketch the graph of f ;
         b. Sketch the graph of $y = x$;
         c. Using the line $y = x$ as the "mirror" reflect the graph of f in it;
         d. Find the domain and range for f and its inverse f^{-1} ;
         e. Find the rule for $f^{-1}(x)$;
         f. Fully define $f^{-1}(x)$.

$y = 3x - 2$

$y = x$

$y = \dfrac{x + 2}{3}$

Point of Intersection
$(1, 1)$

$$dom\ f = [-1,3] \quad ran\ f = [-5,7]$$
$$dom\ f^{-1} = [-5,7] \quad ran\ f^{-1} = [-1,3]$$

To find the rule for the inverse we swap the x and y in the original equation.

$f(x) = 3x - 2$

swap

$x = 3y - 2$

$x + 2 = 3y$

$\dfrac{x+2}{3} = y$ or $y = \dfrac{1}{3}(x+2)$

$\Rightarrow f^{-1}(x) = \dfrac{1}{3}(x+2)$

fully defined : $f^{-1} : [-5,7] \to R,\ f^{-1}(x) = \dfrac{1}{3}(x+2)$

- The domain of $f = $ range f^{-1} and range $f = $ domain f^{-1}
- If the graphs intersect, then the points of intersection MUST also be on the line $y = x$.
- So the points of intersection can be found in 3 ways:
    o $f(x) = f^{-1}(x)$
    o $f(x) = x$
    o $f^{-1}(x) = x$

- It is usually quicker and easier to use one of the last two.
- All functions have inverses, but the inverses may not be functions (they may only be relations).
- e.g. compare $y = x^3$ and $y = x^2$
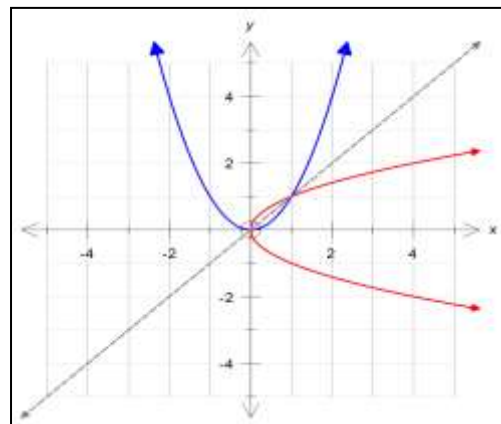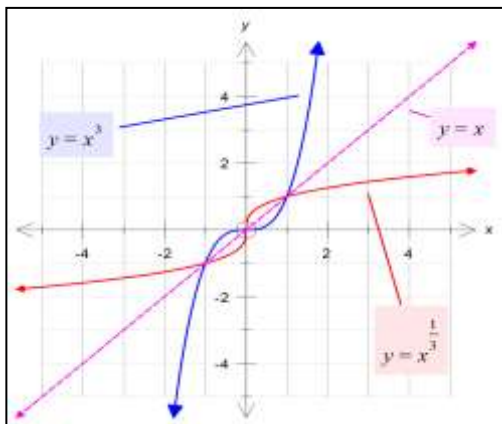
- Original is a function   Original is a function
- Inverse is a function   Inverse is not a function
- *A function f, has an inverse function, written f^{-1} only if f is a one-to-one function.*
- *i.e. a horizontal and a vertical line only crosses the graph of f once.*

- It is possible to restrict the domain on a function, so it will have an inverse function, e.g. $f(x) = x^2$, the domain can be restricted in many ways, e.g $R^+ \cup \{0\}, [2,10), (-\infty, 0], [-5,-1]$

Example: Restrict the domain of $g(x) = x^2 - 3$, so that we have an inverse function $g^{-1}(x)$. Find the two possible $g^{-1}(x)$, where the domain is as large as possible.
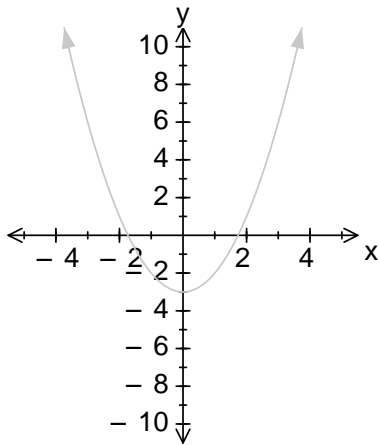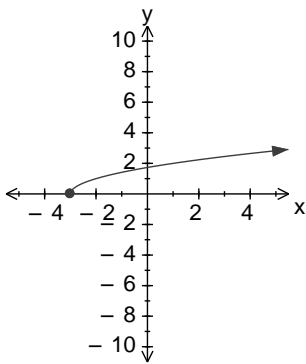

$$\boxed{g(x) = x^2 - 3}$$

g is not $1:1$

$dom\ g = R$

$ran\ g = [-3, \infty)$

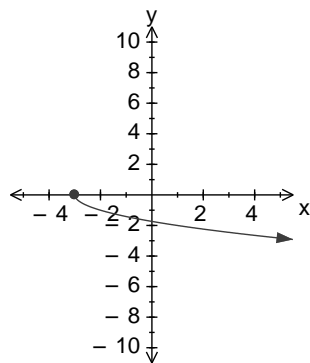- Let's choose the RHS of the curve, i.e $x \geq 0$
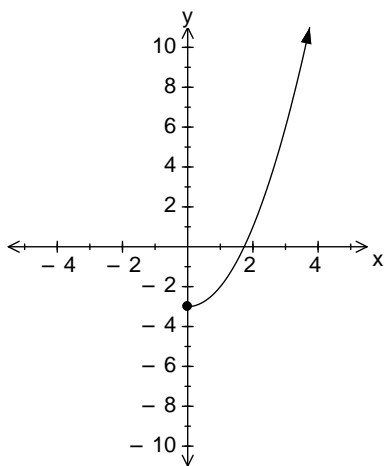
$$g(x) = x^2 - 3$$

$$Let\ y = x^2 - 3$$

$$swap\ x\ \&\ y$$

$$x = y^2 - 3$$

$$x + 3 = y^2$$

$$\pm\sqrt{x+3} = y$$

$$y = +\sqrt{x+3} \quad or \quad y = -\sqrt{x+3}$$

which one is g^{-1}?

$domain: [-3, \infty)$
$range: [0, \infty)$

$domain: [-3, \infty)$
$range: (-\infty, 0]$

- As the $dom\ g = ran\ g^{-1}$ & $ran\ g = dom\ g^{-1}$ then $g^{-1}(x) = +\sqrt{x+3}$.
- $\Rightarrow g^{-1}: [-3, \infty) \to R,\ g^{-1}(x) = \sqrt{x+3}$
- If we used the LHS of the curve, then $\Rightarrow g^{-1}: [-3, \infty) \to R,\ g^{-1}(x) = -\sqrt{x+3}$

Example: If $h: S \rightarrow R, h(x) = \sqrt{x} - 5,$ find:

(a) S (b) h^{-1} (c) $h^{-1}(-2)$ (d) $h^{-1}(-7)$

x Intercept (25, 0)

y Intercept (0, -5)
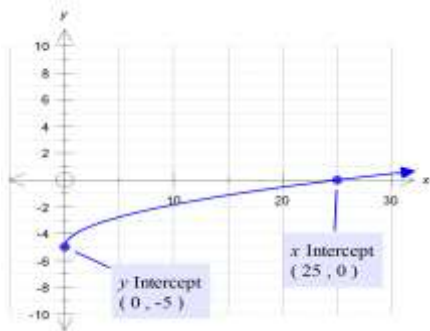
(a) S is the domain, $dom\ h = [0, \infty)$ and $ran\ h = [-5, \infty)$

$$h(x) = \sqrt{x} - 5$$
$$\text{Let } y = \sqrt{x} - 5$$
$$\text{swap } x\ \&\ y$$

(b)
$$x = \sqrt{y} - 5$$
$$x + 5 = \sqrt{y}$$
$$(x+5)^2 = y$$
$$\Rightarrow h^{-1}(x) = (x+5)^2$$
$$h^{-1} : [-5, \infty) \rightarrow R,\ h^{-1}(x) = (x+5)^2$$

(c) $h^{-1}(-2) = ((-2)+5)^2 = (3)^2 = 9$

(d) $h^{-1}(-7) = undefined,\ x = -7\ is\ not\ in\ the\ domain,\ \{-7\} \notin [-5, \infty)$

Example

Find the inverse of the function with rule $f(x) = 3\sqrt{x+2} + 4$ and sketch both functions on one set of axes, clearly showing the exact coordinates of intersection of the two graphs.

Solution

$$y = f(x) = 3\sqrt{x+2} + 4$$
$$\text{Swap } x\ \&\ y$$
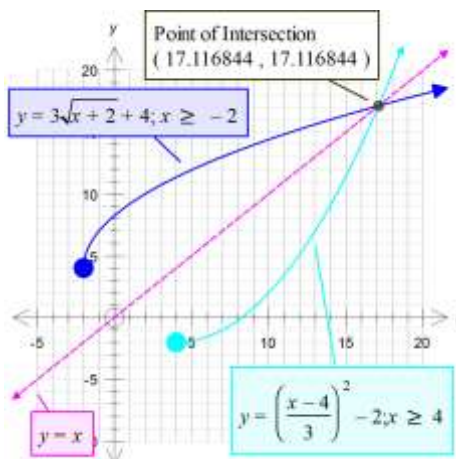$$x = 3\sqrt{y+2} + 4$$
$$\frac{x-4}{3} = \sqrt{y+2}$$
$$\left(\frac{x-4}{3}\right)^2 = y + 2$$
$$\left(\frac{x-4}{3}\right)^2 - 2 = y$$
$$\Rightarrow f^{-1}(x) = \left(\frac{x-4}{3}\right)^2 - 2$$
$$domain\ f^{-1} = ran\ f = [4, \infty)$$

Point of Intersection (17.116844, 17.116844)

$y = 3\sqrt{x+2} + 4; x \geq -2$

$y = \left(\frac{x-4}{3}\right)^2 - 2; x \geq 4$

$y = x$

Note: The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.
The graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by applying the transformation $(x, y) \rightarrow (y, x)$.

In this particular example, it is simpler to solve $f^{-1}(x) = x$ to solve the point of intersection.

- Graphical Calculator can be used to find the inverse of a function
    - Define the function
    - Solve(f(y)=x, y)

- **Ex1F Q 1, 2, 3, 4, 6, 7, 8aceg, 9, 10, 11, 12, 13, 14, 15**

Power Functions

- Power functions are of the form: $f(x) = x^p; p \in Q$ (i.e. p is rational)

Strictly increasing and strictly decreasing functions

- A function f is said to be *strictly increasing* when $a < b$ implies $f(a)<f(b)$ for all a and b in its domain.
- If a function is strictly increasing, then it is a one-to-one function and has an inverse that is also strictly increasing.

Example 1: The function $f : R \to R, f(x) = x^3$ is strictly increasing with zero gradient at the origin.

The inverse function $f^{-1} : R \to R, f^{-1}(x) = x^{\frac{1}{3}}$, is also strictly increasing, with a vertical tangent of undefined gradient at the origin.
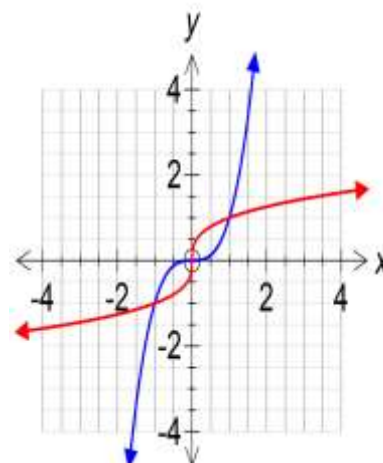
Example 2: The hybrid function g with domain $[0, \infty)$ and rule:

$$g(x) = \begin{cases} x^2 & 0 \le x \le 2 \\ 2x & x > 2 \end{cases}$$ is strictly increasing, and is not
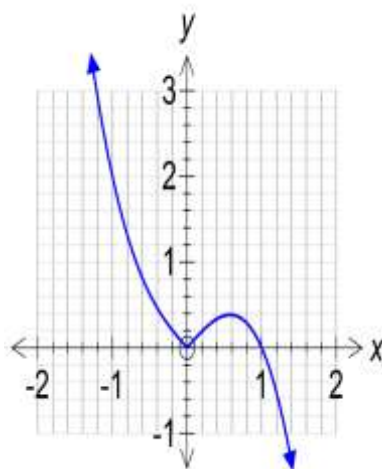
differentiable at $x = 2$.
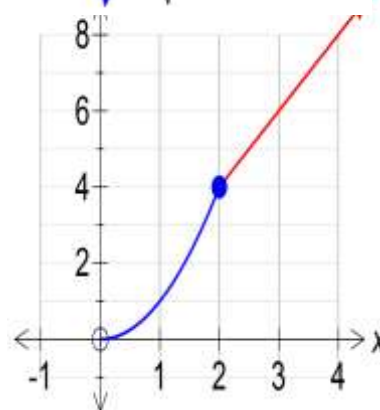
Example 3: Consider
$h : R \to R, h(x) = |x| - x^3$

H is not strictly increasing,
But is strictly increasing over the

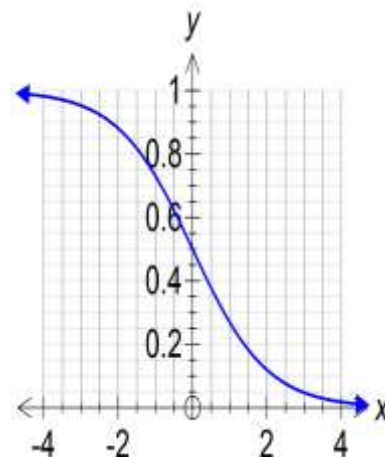$interval \left[0, \dfrac{1}{\sqrt{3}} \right]$.

Strictly Decreasing
A function f is said to be *strictly decreasing* when $a < b$ implies $f(a) > f(b)$ for all a and b in its domain.
A function is said to be strictly decreasing over an interval when $a < b$ implies $f(a) > f(b)$ for all a and b in its interval.
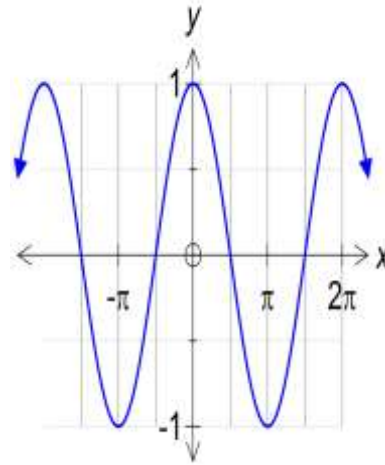
Example 4: The function $f : R \to R, f(x) = \dfrac{1}{e^x + 1}$

The function is strictly decreasing over R.

Example 5: The function $g : R \rightarrow R, g(x) = \cos(x)$
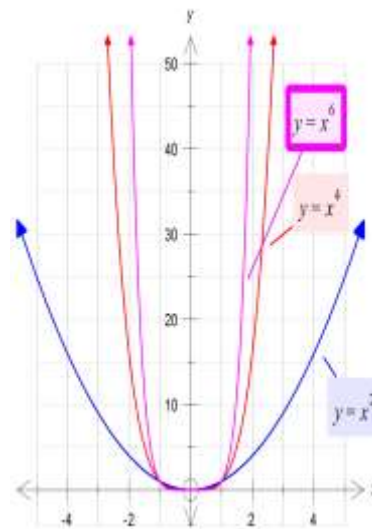
g is not strictly decreasing.

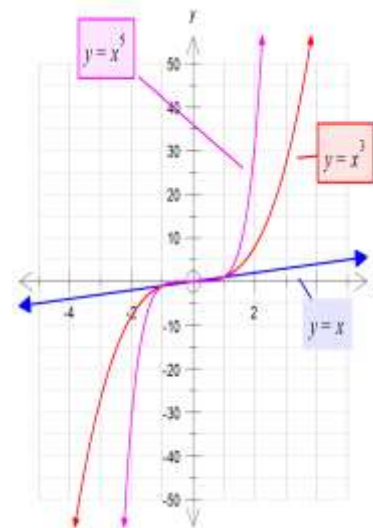But g is strictly decreasing over the interval $[0, \pi]$.

Power functions with positive integer index

- Functions of the form: $f(x) = x^p$; $p = 1, 2, \ldots$
- 2 groups: the even powers and the odd powers.

- Even powers, $f(x) = x^2, x^4, x^6, \ldots$
    - All have the "U-shaped" graph
    - Domain: R
    - Range: $R^+ \cup \{0\}$ or $[0, \infty)$
    - Strictly increasing for $x \geq 0$
    - Strictly decreasing for $x \leq 0$
    - As $x \rightarrow \pm\infty, f(x) \uparrow \infty$

- Odd powers, $f(x) = x, x^3, x^5, \ldots$
    - All slope from bottom left to top right
    - Domain: R
    - Range: R
    - Strictly increasing for all x
    - f is one-to-one
    - As $x \rightarrow +\infty, f(x) \rightarrow \infty$ & $x \rightarrow -\infty, f(x) \rightarrow -\infty$
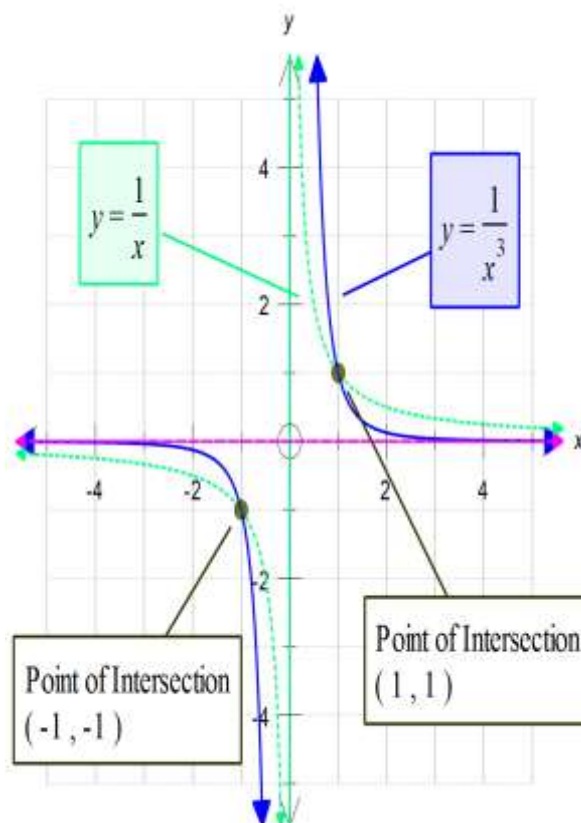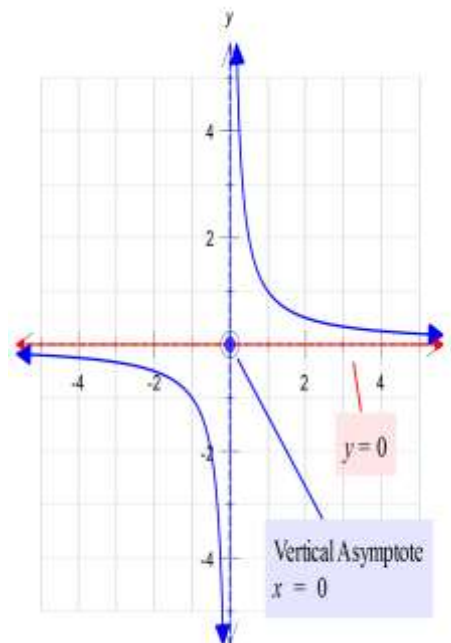
Power functions with negative integer index

- Functions of the form: $f(x) = x^p$; $p = -1, -2, \ldots$
- 2 groups: the even powers and the odd powers.

Odd Negative Powers

Functions of the form: $f(x) = x^p$; $p = -1, -3, \ldots$

- Sketch the graph of $f(x) = \dfrac{1}{x}$ or $= x^{-1}$
- Domain: $R\backslash\{0\}$
- Range: $R\backslash\{0\}$
- Asymptotes:
  - **Horizontal:** $y = 0$

    $As\ x \to \infty, \dfrac{1}{x} \to 0^+$

    $As\ x \to -\infty, \dfrac{1}{x} \to 0^-$
  - **Vertical:** $x = 0$

    $As\ x \to 0^+, \dfrac{1}{x} \to \infty$

    $As\ x \to 0^-, \dfrac{1}{x} \to -\infty$

- Odd function: $f(-x) = \dfrac{1}{-x} = -\dfrac{1}{x}$

- Sketch the graph of $f(x) = \dfrac{1}{x^3}$ or $= x^{-3}$

$y = 0$

Vertical Asymptote
$x = 0$

$y = \dfrac{1}{x}$

$y = \dfrac{1}{x^3}$

Point of Intersection
$(-1, -1)$

Point of Intersection
$(1, 1)$

Even Negative Powers

Functions of the form: $f(x) = x^p; p = -2, -4, \ldots$

- Sketch the graph of $f(x) = \dfrac{1}{x^2}$ or $= x^{-2}$

- Domain: $R\backslash\{0\}$
- Range: $R\backslash\{0\}$
- Asymptotes:
  - **Horizontal:** $y = 0$

    $As\ x \to \infty, \dfrac{1}{x} \to 0^+$

    $As\ x \to -\infty, \dfrac{1}{x} \to 0^-$

  - **Vertical:** $x = 0$

    $As\ x \to 0^+, \dfrac{1}{x} \to \infty$

    $As\ x \to 0^-, \dfrac{1}{x} \to \infty$

- Even function: $f(-x) = \dfrac{1}{(-x)^2} = \dfrac{1}{x^2}$

- Sketch the graph of $f(x) = \dfrac{1}{x^4}$ or $= x^{-4}$

$y = \dfrac{1}{x^2}$

$y = 0$

Vertical Asymptote $x = 0$

$y = \dfrac{1}{x^2}$

Point of Intersection $-(1,1)$

$y = 0$

Point of Intersection $(-1,1)$
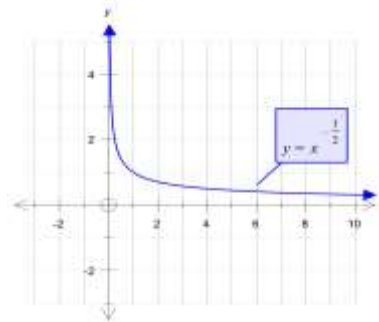
Vertical Asymptote $x = 0$
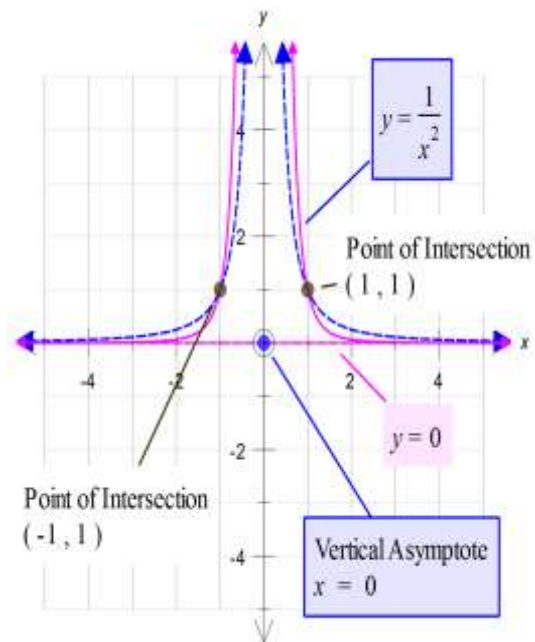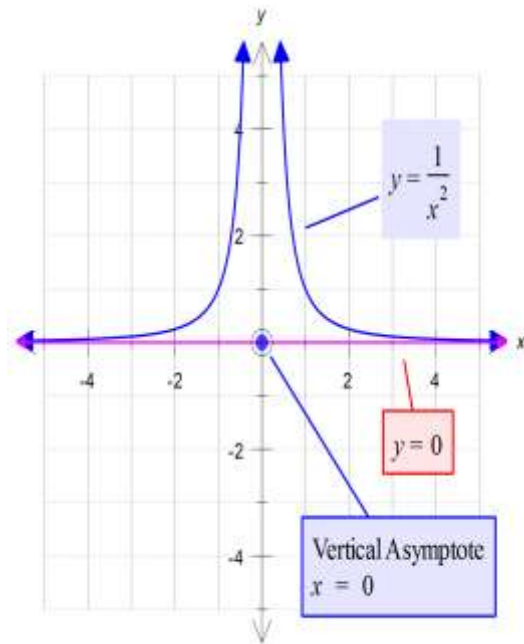
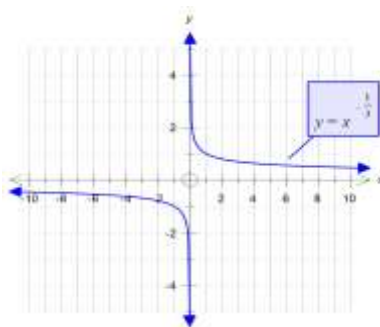# Functions with rational powers: $f(x) = x^{\frac{p}{q}}$

- Of the form: $f(x) = x^{\frac{1}{q}}$

- Remember $f(x) = x^{\frac{1}{q}} = \sqrt[q]{x}$

- Maximal domain is $[0, \infty)$ when q is even:

- Maximal domain is R when q is odd:
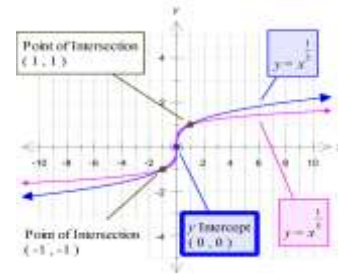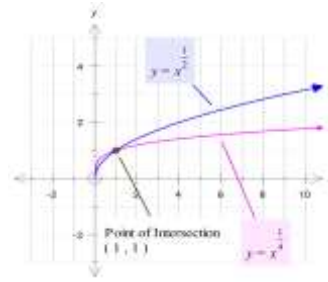
- Consider: $f(x) = \dfrac{1}{\sqrt[q]{x}} = x^{-\frac{1}{q}} = \dfrac{1}{x^{\frac{1}{q}}}$

- Domain $(0, \infty)$ if q is even and $R \backslash \{0\}$ if q is odd.

- $f(x) = \dfrac{1}{\sqrt{x}}$:
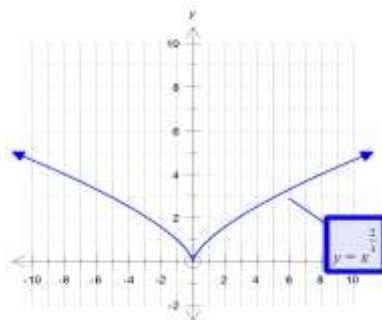
- Asymptotes: $y = 0$ and $x = 0$.
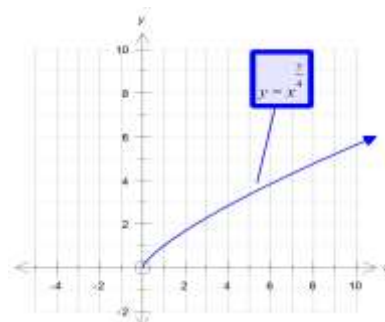
- $f(x) = \dfrac{1}{\sqrt[3]{x}}$

- In General: $f(x) = x^{\frac{p}{q}} = \left(\sqrt[q]{x}\right)^p$

- Always defined for $x \geq 0$, and when q is odd for all x.

- Eg: $f(x) = x^{\frac{2}{3}}$

- e.g.: $f(x) = x^{\frac{3}{4}}$

Inverses of Power Functions

Example: Find the inverse of each of the following:
   a) $f : R \rightarrow R, f(x) = x^5$
   b) $f : (-\infty, 0] \rightarrow R, f(x) = x^4$
   c) $f : R \rightarrow R, f(x) = 8x^3$
   d) $f : (1, \infty) \rightarrow R, f(x) = 64x^6$

$(a) \quad f : R \rightarrow R, f(x) = x^5$

$y = x^5$

Swap x & y

$x = y^5$

$\sqrt[5]{x} = y \quad \text{or} \quad (x)^{\frac{1}{5}} = y$

$f^{-1} : R \rightarrow R, f^{-1}(x) = (x)^{\frac{1}{5}}$

$(b) \quad f : (-\infty, 0] \rightarrow R, f(x) = x^4$

$y = x^4$

Swap x & y

$x = y^4$

$\pm \sqrt[4]{x} = y \quad \text{or} \quad \pm(x)^{\frac{1}{4}} = y$

$\text{ran } f = [0, \infty) = \text{dom } f^{-1}$

$f^{-1} : [0, \infty) \rightarrow R, f^{-1}(x) = -(x)^{\frac{1}{4}}$

$(c) \quad f : R \rightarrow R, f(x) = 8x^3$

$y = 8x^3$

Swap x & y

$x = 8y^3$

$\sqrt[3]{\dfrac{x}{8}} = y \quad \text{or} \quad \left(\dfrac{x}{8}\right)^{\frac{1}{3}} = y = \dfrac{1}{2}x^{\frac{1}{3}}$

$f^{-1} : R \rightarrow R, f^{-1}(x) = \left(\dfrac{x}{8}\right)^{\frac{1}{3}}$

$(d) \quad f : (1, \infty) \rightarrow R, f(x) = 64x^6$

$y = 64x^6$

Swap x & y

$x = 64y^6$

$\pm\sqrt[6]{\dfrac{x}{64}} = y \quad \text{or} \quad \pm\left(\dfrac{x}{64}\right)^{\frac{1}{6}} = y = \pm\dfrac{1}{2}x^{\frac{1}{6}}$

$\text{ran } f = (64, \infty) = \text{dom } f^{-1}$

$f^{-1} : (64, \infty) \rightarrow R, f^{-1}(x) = \dfrac{1}{2}x^{\frac{1}{6}}$

- **Ex1G Q 1, 2, 3, 4, 5**

Past VCAA Exam Questions
2008

Question 6

The simultaneous linear equations

$$ax + 3y = 0$$
$$2x + (a + 1)y = 0$$

where a is a real constant, have infinitely many solutions for

A. $a \in R$
B. $a \in \{-3, 2\}$
C. $a \in R \setminus \{-3, 2\}$
D. $a \in \{-2, 3\}$
E. $a \in R \setminus \{-2, 3\}$

B

For infinitely many solutions the lines must have equivalent equations.
$ax + 3y = 0$
$2x + (a + 1)y = 0$
Since the y-intercept for both equations is the same, i.e. zero, then the values of a can be found by letting the determinant of the corresponding matrix equal to zero and solving for a, without checking the solutions for parallel lines.

$$\begin{vmatrix} a & 3 \\ 2 & a+1 \end{vmatrix} = 0$$

Hence $a = -3$ or $a = 2$. Alternatively, using multiples of coefficients and equations leads to $a(a+1) = 6$. Hence $a = -3$ or $a = 2$.

23% students found the values a for which there was a unique solution.

Question 7

The inverse of the function $f: R^+ \to R, \ f(x) = \dfrac{1}{\sqrt{x}} - 3$ is

A. $f^{-1}: R^+ \to R \qquad f^{-1}(x) = (x+3)^2$

B. $f^{-1}: R^+ \to R \qquad f^{-1}(x) = \dfrac{1}{x^2} + 3$

C. $f^{-1}: (3, \infty) \to R \qquad f^{-1}(x) = \dfrac{-1}{(x-3)^2}$

D. $f^{-1}: (-3, \infty) \to R \qquad f^{-1}(x) = \dfrac{1}{(x+3)^2}$

E. $f^{-1}: (-3, \infty) \to R \qquad f^{-1}(x) = -\dfrac{1}{x^2} - 3$

D

Question 20

The function $f: B \to R$ with rule $f(x) = 4x^3 + 3x^2 + 1$ will have an inverse function for

A. $B = R$

B. $B = \left(\dfrac{1}{2}, \infty\right)$

C. $B = \left(-\infty, \dfrac{1}{2}\right]$

D. $B = \left(-\infty, \dfrac{1}{2}\right)$

E. $B = \left[-\dfrac{1}{2}, \infty\right)$

B

2009

Question 1

The simultaneous linear equations

$$kx - 3y = 0$$
$$5x - (k + 2)y = 0$$

where k is a real constant, have a unique solution provided

A. $k \in \{-5, 3\}$
B. $k \in R \setminus \{-5, 3\}$
C. $k \in \{-3, 5\}$
D. $k \in R \setminus \{-3, 5\}$
E. $k \in R \setminus \{0\}$

B

Using multiples of coefficients and combining equations, a unique solution exists when $k(k+2) - 15$ is non zero, that is, $k \in R \setminus \{-5, 3\}$.

Alternatively, solve $\begin{vmatrix} k & -3 \\ 5 & -k-2 \end{vmatrix} = 0$

$k = -5$ or $k = 3$ (infinite number of solutions).
A unique solution will occur if $k \in R \setminus \{-5, 3\}$.

Question 3

Let $f: R \setminus \{0\} \to R$ where $f(x) = \dfrac{3}{x} - 4$. Find f^{-1}, the inverse function of f.

_____

_____

Question 3

Marks	0	1	2	3	Average
%	8	10	44	38	2.1

$f^{-1}: R \setminus \{-4\} \to R$ where $f^{-1}(x) = \dfrac{3}{x + 4}$

Few students realised that the inverse function, f^{-1}, required the rule and the domain to be specified. The definition of f should have provided students with a clue.

2010

Question 7

The simultaneous linear equations $(m-1)x + 5y = 7$ and $3x + (m-3)y = 0.7m$ have infinitely many solutions for

A. $m \in R \setminus \{0, -2\}$
B. $m \in R \setminus \{0\}$
C. $m \in R \setminus \{6\}$
D. $m = 6$
E. $m = -2$

Q7 $(m-1)x + 5y = 7$ and $3x + (m-3)y = 0.7m$

Change to standard form:

$y = -\dfrac{m-1}{5}x + \dfrac{7}{5}$ and $y = -\dfrac{3}{m-3}x + \dfrac{0.7m}{m-3}$

Equate coefficients: $-\dfrac{m-1}{5} = -\dfrac{3}{m-3}$ and $\dfrac{7}{5} = \dfrac{0.7m}{m-3}$
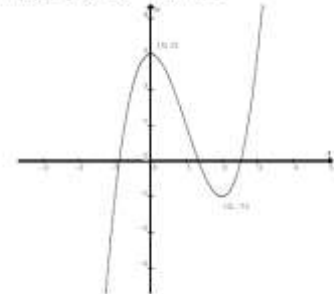
$\therefore m = 6$

D

Question 9

The function $f: (-\infty, a] \to R$ with rule $f(x) = x^3 - 3x^2 + 3$ will have an inverse function provided

A. $a \le 0$
B. $a \ge 2$
C. $a \ge 0$
D. $a \le 2$
E. $a \le 1$

Q9 The graph of $f(x) = x^3 - 3x^2 + 3$:

The graph is 1 to 1 in the interval $(-\infty, 0]$, $\therefore a \le 0$

A

Question 5

For the system of simultaneous linear equations

$$x = 5$$
$$z + y = 10$$
$$z - y = 6$$

an equivalent matrix equation is

A. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$

C. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix}$

Q5

$1x + 0y + 0z = 5$
$0x + 1y + 1z = 10$
$0x - 1y + 1z = 6$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$

B

Question 3

Let $f: R^+ \to R$ where $f(x) = \dfrac{1}{x^2}$.

a. Find $g(x) = f(f(x))$.

_____
_____
_____
_____
_____

b. Evaluate $g^{-1}(16)$, where g^{-1} is the inverse function of g.

_____
_____
_____

1 mark

Q3a $f(x) = \dfrac{1}{x^2}$, $g(x) = f(f(x)) = \dfrac{1}{\left(\frac{1}{x^2}\right)^2} = x^4$, $x \in R^+$

Q3b $g^{-1}(x) = x^{\frac{1}{4}}$, $x \in R^+$, $g^{-1}(16) = 16^{\frac{1}{4}} = \left(2^4\right)^{\frac{1}{4}} = 2$

Question 1

The midpoint of the line segment joining $(0, -5)$ to $(d, 0)$ is

A. $\left(\dfrac{d}{2}, -\dfrac{5}{2}\right)$

B. $(0, 0)$

C. $\left(\dfrac{d-5}{2}, 0\right)$

D. $\left(0, \dfrac{5-d}{2}\right)$

E. $\left(\dfrac{5+d}{2}, 0\right)$

Question 2

The gradient of a line perpendicular to the line which passes through $(-2, 0)$ and $(0, -4)$ is

A. $\dfrac{1}{2}$

B. -2

C. $-\dfrac{1}{2}$

D. 4

E. 2

Question 3

If $x + a$ is a factor of $4x^3 - 13x^2 - ax$, where $a \in \mathbb{R}\backslash\{0\}$, then the value of a is

A. -4

B. -3

C. -1

D. 1

E. 2

Question 5

The inverse function of $g: [2, \infty) \to R, g(x) = \sqrt{2x-4}$ is

A. $g^{-1}: [2, \infty) \to R, g^{-1}(x) = \dfrac{x^2+4}{2}$

B. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = (2x-4)^2$

C. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \sqrt{\dfrac{x}{2}+4}$

D. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \dfrac{x^2+4}{2}$

E. $g^{-1}: R \to R, g^{-1}(x) = \dfrac{x^2+4}{2}$

Question 8

Consider the function $f: R \to R, f(x) = x(x-4)$ and the function

$$g: \left[\dfrac{3}{2}, 5\right) \to R, g(x) = x + 3.$$

If the function $h = f + g$, then the domain of the inverse function of h is

A. $[0, 13)$

B. $\left[-\dfrac{3}{4}, 10\right]$

C. $\left(-\dfrac{3}{4}, \dfrac{15}{4}\right]$

D. $\left[\dfrac{3}{4}, 13\right)$

E. $\left[\dfrac{3}{2}, 13\right)$

Question 18

The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when

A. $-7 < w < 25$

B. $w \le -7$

C. $w \ge 25$

D. $w < -7$ or $w > 25$

E. $w > 1$

Q1 Midpoint: $\left(\dfrac{0+d}{2}, \dfrac{-5+0}{2}\right) = \left(\dfrac{d}{2}, -\dfrac{5}{2}\right)$ A

Q2 Gradient of given line: $\dfrac{0-(-4)}{(-2)-0} = -2$

∴ gradient of perpendicular line is $\dfrac{1}{2}$.   A

Q3 $f(x) = 4x^3 - 13x^2 - ax$ where $a \neq 0$

$f(-a) = -4a^3 - 13a^2 + a^2 = 0$, ∴ $-4a^3 - 12a^2 = 0$

$-4a^2(a+3) = 0$, ∴ $a = -3$ B

Q5 The range of $y = \sqrt{2x-4}$ is $[0, \infty)$ and it is the domain of the inverse. Equation of the inverse: $x = \sqrt{2y-4}$, ∴ $y = \dfrac{x^2+4}{2}$

∴ the rule is $g^{-1}(x) = \dfrac{x^2+4}{2}$ D

Q8 $h(x) = f(x) + g(x) = x(x-4) + x + 3 = x^2 - 3x + 3$ and

$x \in \left[\dfrac{3}{2}, 5\right)$. The range of h is $\left[\dfrac{3}{4}, 13\right)$, ∴ the domain of h^{-1} is
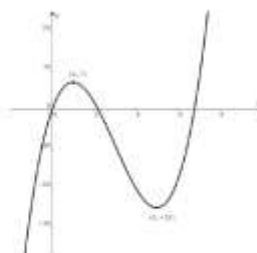
$\left[\dfrac{3}{4}, 13\right)$.   D

Q18 Consider the graph of $f(x) = x^3 - 9x^2 + 15x$.

The local max and min are $(1, 7)$ and $(5, -25)$ respectively. To Pass through the x-axis once, the graph must be lowered by more than 7 units or lifted up by more than 25 units.

∴ $w < -7$ or $w > 25$.   D

Question 4

If the function f has the rule $f(x) = \sqrt{x^2 - 9}$ and the function g has the rule $g(x) = x + 5$

a. find integers c and d such that $f(g(x)) = \sqrt{(x+c)(x+d)}$

_____

_____

_____

_____

_____

_____

b. state the maximal domain for which $f(g(x))$ is defined.

_____

_____

_____

_____

_____

2 marks

> Q4a $f(x) = \sqrt{x^2 - 9}$, $g(x) = x + 5$,
>
> $f(g(x)) = \sqrt{(x+5)^2 - 9} = \sqrt{(x+2)(x+8)}$
>
> $\therefore c = 2$ and $d = 8$ OR $c = 8$ and $d = 2$
>
> Q4b $(x+2)(x+8) \geq 0$, $\therefore x \leq -8$ OR $x \geq -2$

Question 6

Consider the simultaneous linear equations

$$kx - 3y = k + 3$$
$$4x + (k+7)y = 1$$

where k is a real constant.

a. Find the value of k for which there are infinitely many solutions.

> Q6a $kx - 3y = k + 3$, $\therefore y = \dfrac{k}{3}x + \dfrac{-k-3}{3}$
>
> $4x + (k+7)y = 1$, $\therefore y = \dfrac{-4}{k+7}x + \dfrac{1}{k+7}$
>
> Infinitely many solutions: $\dfrac{k}{3} = \dfrac{-4}{k+7}$ and $\dfrac{-k-3}{3} = \dfrac{1}{k+7}$
>
> $\therefore k^2 + 7k + 12 = 0$ and $k^2 + 10k + 24 = 0$
>
> $\therefore (k+3)(k+4) = 0$ and $(k+6)(k+4) = 0$
>
> $\therefore$ only $k = -4$ satisfies both constraints.

3 marks

b. Find the values of k for which there is a unique solution.

> Q6b Unique solution: $\dfrac{k}{3} \neq \dfrac{-4}{k+7}$, $\therefore (k+3)(k+4) \neq 0$
>
> $\therefore k \neq -3, -4$, i.e. $k \in R \backslash \{-4, -3\}$
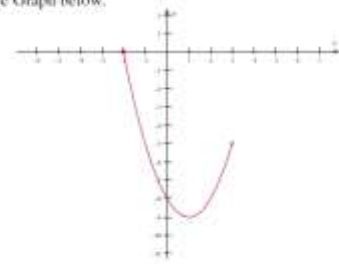
1 mark

Question 3

The range of the function $f:[-2, 3) \to R$, $f(x) = x^2 - 2x - 8$ is

A. R

B. $(-9, -5]$

C. $(-5, 0)$
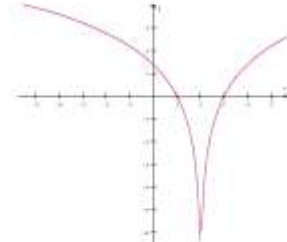
D. $[-9, 0]$

E. $[-9, -5)$

Q3 See Graph below.

D

Question 5

Let the rule for a function g be $g(x) = \log_e((x-2)^2)$. For the function g, the

A. maximal domain $= R^+$ and range $= R$

B. maximal domain $= R\backslash\{2\}$ and range $= R$

C. maximal domain $= R\backslash\{2\}$ and range $= (-2, \infty)$

D. maximal domain $= [2, \infty)$ and range $= (0, \infty)$

E. maximal domain $= [2, \infty)$ and range $= [0, \infty)$

Q5 $g(x) = \log_e((x-2)^2)$, $(x-2)^2 > 0$, $\therefore x \neq 2$

See graph below.

B

Question 17

A system of simultaneous linear equations is represented by the matrix equation

$$\begin{bmatrix} m & 3 \\ 1 & m+2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}.$$

The system of equations will have **no solution** when

A. $m = 1$

B. $m = -3$

C. $m \in \{1, -3\}$

D. $m \in R\backslash\{1\}$

E. $m \in \{1, 3\}$

Q17 Change the matrix equation to two simultaneous linear equations, writing y as the subject of the equations.

$$y = -\frac{m}{3}x + \frac{1}{3} \quad \text{and} \quad y = -\frac{1}{m+2}x + \frac{m}{m+2}$$

The equation will have no solution when:

$$\frac{m}{3} = \frac{1}{m+2} \quad \text{AND} \quad \frac{1}{3} \neq \frac{m}{m+2}$$

i.e. $m^2 + 2m - 3 = 0$, $(m-1)(m+3) = 0$ AND $m \neq 1$

$\therefore m = -3$

B

Question 3

The rule for function h is $h(x) = 2x^3 + 1$. Find the rule for the inverse function h^{-1}.

Q3 $h(x) = 2x^3 + 1$, equation of the inverse is $x = 2y^3 + 1$,

$\therefore y = \left(\dfrac{x-1}{2}\right)^{\frac{1}{3}}$, $\therefore h^{-1}(x) = \left(\dfrac{x-1}{2}\right)^{\frac{1}{3}}$

2 marks

Question 2

The midpoint of the line segment that joins $(1, -5)$ to $(d, 2)$ is

A. $\left(\frac{d+1}{2}, -\frac{3}{2}\right)$

B. $\left(\frac{1-d}{2}, -\frac{7}{2}\right)$

C. $\left(\frac{d-4}{2}, 0\right)$

D. $\left(0, \frac{1-d}{3}\right)$

E. $\left(\frac{5+d}{2}, 2\right)$

Q2 Midpoint is $\left(\dfrac{d+1}{2}, \dfrac{2+(-5)}{2}\right) = \left(\dfrac{d+1}{2}, -\dfrac{3}{2}\right)$.

A

Q3 $7(-a)^3 + 9(-a)^2 - 5a(-a) = 0$, $a \neq 0$

$-7a^3 + 14a^2 = 0$, $-7a^2(a-2) = 0$, $\therefore a = 2$

E

Question 3

If $x + a$ is a factor of $7x^3 + 9x^2 - 5ax$, where $a \in R\backslash\{0\}$, then the value of a is

A. -4

B. -2

C. -1

D. 1

E. 2

Question 5

If $f: (-\infty, 1) \to R$, $f(x) = 2\log_e(1-x)$ and $g: [-1, \infty) \to R$, $g(x) = 3\sqrt{x+1}$, then the maximal domain of the function $f + g$ is

A. $[-1, 1)$

B. $(1, \infty)$

C. $(-1, 1]$

D. $(-\infty, -1]$

E. R

Q5 The maximal domain of $f + g$ is the intersection of the domains of f and g, i.e. $[-1, 1)$.

A

Question 21

The cubic function $f: R \to R$, $f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, has no stationary points when

A. $c > \dfrac{b^2}{4a}$

B. $c < \dfrac{b^2}{4a}$

C. $c < 4b^2a$

D. $c > \dfrac{b^2}{3a}$

E. $c < \dfrac{b^2}{3a}$

Q21 $f(x) = ax^3 - bx^2 + cx$, $f'(x) = 3ax^2 - 2bx + c$

f has no stationary points, $\therefore f'(x) \neq 0$, $\therefore \Delta = B^2 - 4AC < 0$

$\therefore 4b^2 - 12ac < 0$, $\therefore c > \dfrac{b^2}{3a}$ 　　　　D

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Question 2

The linear function $f: D \to R$, $f(x) = 4 - x$ has range $[-2, 6)$.
The domain D of the function is

A. $[-2, 6)$
B. $(-2, 2]$
C. R
D. $(-2, 6]$
E. $[-6, 2]$

Q2 $y = 4 - x$

When $y = -2$, $-2 = 4 - x$, $x = 6$

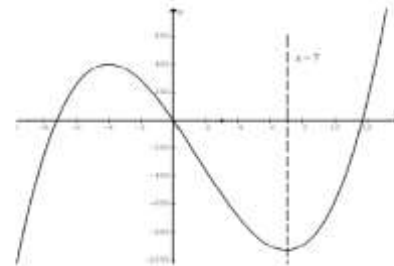When $y = 6$, $6 = 4 - x$, $x = -2$

The domain is $(-2, 6]$ 　　　　D

Question 6

The function $f: D \to R$ with rule $f(x) = 2x^3 - 9x^2 - 168x$ will have an inverse function for

A. $D = R$
B. $D = (7, \infty)$
C. $D = (-4, 8)$
D. $D = (-\infty, 0)$
E. $D = \left[-\dfrac{1}{2}, \infty\right)$

Q6

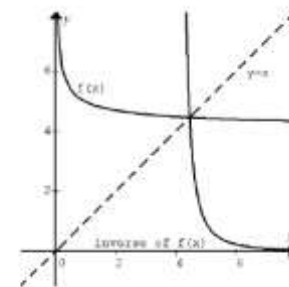f is a one-to-one function in the interval $(7, \infty)$. 　　　　B

Question 9

The inverse of the function $f: R^+ \to R$, $f(x) = \dfrac{1}{\sqrt{x}} + 4$ is

A. $f^{-1}: (4, \infty) \to R$ 　　 $f^{-1}(x) = \dfrac{1}{(x-4)^2}$

B. $f^{-1}: R^+ \to R$ 　　 $f^{-1}(x) = \dfrac{1}{x^2} + 4$

C. $f^{-1}: R^+ \to R$ 　　 $f^{-1}(x) = (x+4)^2$

D. $f^{-1}: (-4, \infty) \to R$ 　　 $f^{-1}(x) = \dfrac{1}{(x+4)^2}$

E. $f^{-1}: (-\infty, 4) \to R$ 　　 $f^{-1}(x) = \dfrac{1}{(x-4)^2}$

Q9

Domain of the inverse is $(4, \infty)$ 　　　　A

Question 17

The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have no solution for

A. $a = 3$
B. $a = -3$
C. both $a = 3$ and $a = -3$
D. $a \in R \backslash \{3\}$
E. $a \in R \backslash \{-3, 3\}$

Q17 $y = \dfrac{a}{3}x - \dfrac{5}{3}$ and $y = \dfrac{3}{a}x - \left(\dfrac{8}{a} - 1\right)$

To have no solution (intersection), the gradient must be the same
and the y-intercept must be different, i.e. $\dfrac{a}{3} = \dfrac{3}{a}$ and $\dfrac{8}{a} - 1 \neq \dfrac{5}{3}$,

$\therefore a = -3$ 　　　　B

Question 18

The graph of $y = kx - 4$ intersects the graph of $y = x^2 + 2x$ at two distinct points for

A. $k = 6$
B. $k > 6$ or $k < -2$
C. $-2 \leq k \leq 6$
D. $6 - 2\sqrt{3} \leq k \leq 6 + 2\sqrt{3}$
E. $k = -2$

Q18 $y = x^2 + 2x$ and $y = kx - 4$

$\therefore x^2 + 2x = kx - 4$, $x^2 + (2-k)x + 4 = 0$

For 2 intersections, $\Delta > 0$, $(2-k)^2 - 4(1)(4) > 0$
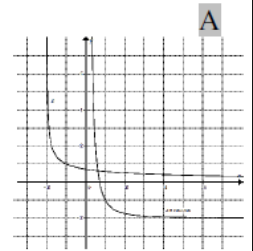
$\therefore k < -2$ or $k > 6$ 　　　　B

Question 2

The inverse function of $f:(-2, \infty) \to R$, $f(x)=\dfrac{1}{\sqrt{x+2}}$ is

A. $f^{-1}: R^+ \to R$ $f^{-1}(x)=\dfrac{1}{x^2}-2$

B. $f^{-1}: R\setminus\{0\} \to R$ $f^{-1}(x)=\dfrac{1}{x^2}-2$

C. $f^{-1}: R^+ \to R$ $f^{-1}(x)=\dfrac{1}{x^2}+2$

D. $f^{-1}: (-2, \infty) \to R$ $f^{-1}(x)=x^2+2$

E. $f^{-1}: (2, \infty) \to R$ $f^{-1}(x)=\dfrac{1}{x^2-2}$
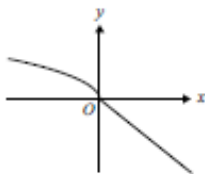
Q2   A

$$x=\dfrac{1}{\sqrt{y+2}}, \quad y=\dfrac{1}{x^2}-2, \therefore f^{-1}(x)=\dfrac{1}{x^2}-2$$

Question 5

Part of the graph of $y=f(x)$ is shown below.

The corresponding part of the graph of the inverse function $y=f^{-1}(x)$ is best represented by

A.

B.

C.

D.

E.

Q5   Reflection in the line $y=x$ E

Question 6

For the polynomial $P(x) = x^3 - ax^2 - 4x + 4$, $P(3) = 10$, the value of a is

A. -3
B. -1
C. 1
D. 3
E. 10

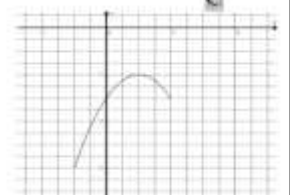**Question 7**

The range of the function $f: (-1, 2] \to R$, $f(x) = -x^2 + 2x - 3$ is

A. R
B. $(-6, -3]$
C. $(-6, -2]$
D. $[-6, -3]$
E. $[-6, -2]$

Q6 $P(3) = 3^3 - a.3^2 - 4(3) + 4 = 10, \therefore a = 1$ C

Q7   C

Question 17

A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts.

The set of all possible values of c is

A. R

B. R^+

C. $(0, 4)$

D. $(0, 4)$

E. $(-\infty, 4)$

Q17 By CAS $y = x^3 - 3x^2$ has a local max at $(0, 0)$ and a local min at $(2, -4)$. To have three distinct x-intercepts, $y = x^3 - 3x^2$ must translate upwards by less than 4 units. Hence $c \in (0, 4)$.   **D**

Question 21

The graphs of $y = mx + c$ and $y = ax^2$ will have no points of intersection for all values of m, c and a such that

A. $a > 0$ and $c > 0$

B. $a > 0$ and $c < 0$

C. $a > 0$ and $c > -\dfrac{m^2}{4a}$

D. $a < 0$ and $c > -\dfrac{m^2}{4a}$

E. $m > 0$ and $c > 0$

Q21 Let $ax^2 = mx + c$, $ax^2 - mx - c = 0$

No intersections: $\Delta < 0$, $(-m)^2 - 4a(-c) < 0$, $m^2 + 4ac < 0$

If $a > 0$, $c < -\dfrac{m^2}{4a}$; if $a < 0$, $c > -\dfrac{m^2}{4a}$ **D**