Five Ways to Solve Proportion Problems

Understanding ratios and using proportional thinking is the most important set of math concepts we teach in middle school. Ratios grow out of fractions and lead into linear functions.

Proportional thinking is used all the time in everyday life – unit prices, miles per gallon, converting measurement units, etc. etc. etc.

Types of ratios: Part-to-part, part-to-whole, unit rates

We often use "part to part" ratios for describing everyday situations, such as *At the school dance, there were 4 boys for every 3 girls.*

But they can be tricky to work with for middle school students. They are not "part to whole" ratios, which are just another way of talking about fractions. Given the situation above, we might pose the problem: *If there were 133 students at the dance, how many were boys and how many were girls?* Student might try to use a fraction like 4/3 or 3/4 to solve this problem, but part to part ratios are not fractions.

To help students understand how to work with a problem like this, ask them to draw a picture of what 4 to 3 looks like. They might draw something like this:

The nice aspect of this drawing is that it points out that there are 7 total students in each group of 4 boys and 3 girls. So 3/7 are boys and 4/7 are girls. Of course there are many other ways to solve this than multiplying by the fraction.

A part-to-whole situation might be something like this: *In a bag of M&Ms that contains 40 candies, 8 are red. Every bag, no matter how large, has this same ratio of red candies. How many red candies would you expect to find in a bag of 25 M&Ms?*

In this problem, the fraction of candies that are red is evident from the problem. Again, there are many ways to solve it, as described in the "Five Ways" below.

## Constant of proportionality

Ratios can be expressed as equations using a constant of proportionality, the number that multiplies one part to get the other. For the M&M problem above,

- Let x = the total number of candies
- Let y = the number of red candies

What equation relates y and x? $y = \frac{1}{5}x$ The constant of proportionality is $\frac{1}{5}$.

A table of values can be written from this equation, which is sometimes known as a ratio table. The table shows the multiplicative thinking involved with ratios. If the equation is rewritten as $5y = x$, it is clear that the number of red candies is multiplied by 5 to get the total number of candies. For every increase of 1 red candy, there are 5 times as many total candies.

A graph of the original equation looks like this, clearly showing the connection between the slope of the graph and the constant of proportionality.
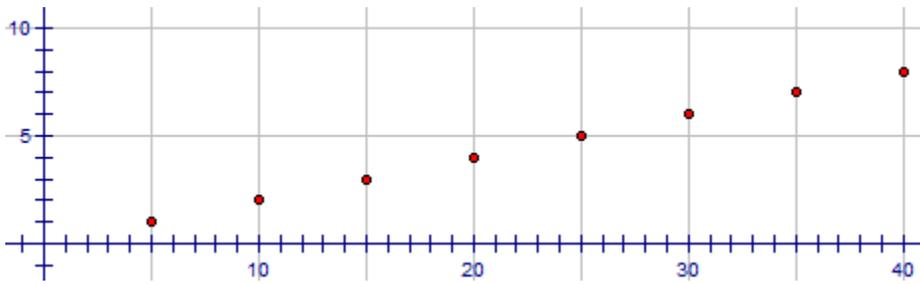
For every 5 M&Ms, there is 1 red M&M.

Five Ways to Solve Proportion Problems

Proportional thinking is very useful for solving many types of problems.

> *If 3 pounds of hamburger feeds 12 people, how many pounds are needed for 30 people?*
>
> *If we can drive 150 miles in 3 hours, how many hours will it take us to drive 400 miles?*

Either of these problems can be solved by finding a "unit rate" and then multiplying or dividing to find the answer (Strategy 1). If 3 pounds of hamburger feeds 12 people, then 1 pound feeds 4 people, and 7 ½ pounds are needed for 30 people (4 times what equals 30?). If we can drive 150 miles in 3 hours, we can drive 50 miles in 1 hour, so it will take us 8 hours to drive 400 miles. (50 times what equals 400?)

> Note: The ability to see situations involving factors and products as either multiplication or division is critical for knowing how to solve these problems. This is an important aspect of number sense. It can be reinforced by solving area-of-rectangle problems where either both sides are known or the area is known and only one side. Or just using triangle flash cards or related number talks to develop this multiplicative thinking.

The unit rate is the "constant of proportionality," which will be explained in the section on graphing. As a preview, the unit rate in the hamburger problem is 4 people per one pound. An equation can be written that relates people (y) to pounds (x): $y = 4x$. Take any number of pounds, multiply it by 4, and we get the number of people we can feed. 4 is the constant of proportionality.

In the driving problem, the unit rate is 50 miles in one hour (50 mph). The equation relating number of hours (t) to miles (d) is $d = 50t$. 50 is the constant of proportionality, the unit rate, and the slope of the graph of the linear equation.

These two proportion problems can also be solved by writing a ratio table (Strategy 2). For the hamburger problem:

Pounds	3	6	9	12
People	12	24	36	48

The number of people is 30, halfway between 24 and 36, so the number of pounds is halfway between 6 and 9. If a student is having difficulty figuring this out, have them use a number line and "count in" until they get to the middle. The answer is halfway between 7 and 8.
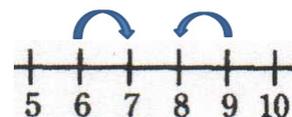
For the driving problem:

Miles	50	150	200	250	300	350	400	450
Hours	1	3			6			9

The answer is obvious from the table, after including columns for each 50 mile increment.

Any ratio table can be drawn vertically instead of horizontally. As a vertical table, it connects to tables of values used to graph relationships. In 6th grade, students simply graph these relationships. In 7th grade, they relate the unit rate to the slope of the graph and to the constant of proportionality.

Pounds	3	6	9
People	12	24	36

Pounds	People
3	12
6	24
9	36

This table corresponds to the ordered pairs (3,12), (6,24), (9, 36).

A third way to solve proportion problems is by using a method called a tape diagram, shown below for this problem:

David and Jason are doing yard work as a summer job. For every 2 hours David works, Jason works 3 hours. One week they worked 35 hours altogether. How many hours did each boy work?
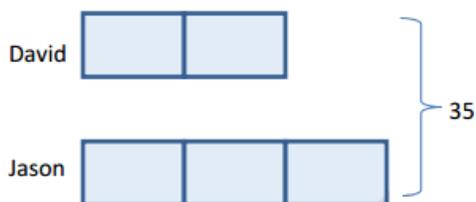

The ratio is shown by drawing 2 boxes (a "tape" that is 2 boxes long) for David and a "tape" that is 3 boxes long for Jason (boxes must be the same size). When these 5 boxes altogether represent the total of 35 hours of work, how much does each box represent? (7) Then how many hours did David work? (14) How many hours did Jason work? (21) To check, does 14 + 21 = 35?

The tape diagram is very much like the bar model from Singapore Math. It is useful for problems that involve part to part ratios, where the units are the same.

Here's another problem where one part is known rather than the whole:

The ratio of children to adults at the football game was 2:3. If there were 140 children at the football game, how many adults were there?
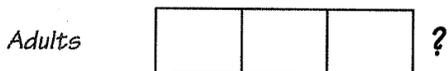
Do you see how to solve this? If two boxes represent 140, then each box represents 70. Three boxes for the adults represents 3x70 = 210.

The fourth way to solve proportion problems is to set them up as an equality between the two ratios.

Proportion equations are a class of linear equations that falls in the learning progression around 7th grade (some are 2-step problems), and should be solved as a linear equation rather than using the algorithm of "cross multiplication."

A proportion equation can be used to represent this problem:

> *There are 8 red M&Ms in every package of 40 M&Ms. If this ratio is used for all size bags, how many red M&Ms would be in a big bag with 200 M&Ms?*

The key to help students set up this equation is in the language of the problem: "*If this ratio is used for all...*" This means that the ratio in the first bag of 8:40 is the same as (equals) the ratio in the second bag of *x*:200. Both of these ratios are part to whole; both represent actual fractions.

$$\frac{8}{40} = \frac{x}{200}$$ Multiply both sides by 200 to "isolate the *x*." This gives:

$$200 \cdot \frac{8}{40} = \frac{x}{200} \cdot 200 \quad \text{or} \quad \frac{1600}{40} = x$$ Carry out the division on the left side.

$$40 = x$$

Note: I strongly recommend never teaching cross-multiplication. Many students see it as just one more procedure without meaning, and often use it wrongly to solve equations like the one in the box at right. They "cross multiply" the two fractions on the left side, winding up with the second equation, which of course is wrong. It's not any easier to solve the above equation by cross multiplying – it is still a two-step equation.

$$\frac{3}{4} \times \frac{x}{2} = \frac{9}{8}$$

$$6 \times 4x = \frac{9}{8}$$

The type of proportion equation that results from the M&M problem seems easier to solve for many students because the *x* does not have to be "moved" to solve this. The type of proportion equation where the variable is in the denominator can seem more confusing to some students, as in the problem about children and adults at a football game. If the equation is written as the problem is presented, it would say 2 children is to 3 adults as 140 children is to *x* adults:

$$\frac{2}{3} = \frac{140}{x}$$

However, if the problem is read a little differently, it can be set up to put the unknown in the numerator: 3 adults is to 2 children as *x* adults is to 140 children; 3:2 = *x*:140

$$\frac{3}{2} = \frac{x}{140}$$

The reason it doesn't matter which quantity is the numerator and which is the denominator is that it's the ratio that matters and always stays the same. What is important is that the proportion equation is written with one quantity always on top (in this case, time) and the other quantity always on bottom (in this case, distance). For the M&M problem, "red M&Ms" were always written first, and "total M&Ms" were always written second: 8:40 as *x*:200.

The type of problem like *David and Jason are doing yard work* is difficult to set up as a proportion equation because the first set of information gives a part to part ratio (*For every 2 hours David works, Jason works 3 hours*) but the question asks for two part to whole ratios (David's part and Jason's part in a 35 hour week). Proportion equations must be set up to represent the same ratios, either part to part or

part to whole. Since 2:3 is part to part, we could use the following equation, where D stands for the actual hours worked by David in the 35 hour week:

$$\frac{2}{3} = \frac{D}{J} \quad \text{and} \quad J+D = 35, \quad \text{so} \quad J = 35\text{-}D$$

$$\frac{2}{3} = \frac{D}{35 - D} \quad \text{where 35-}D \text{ represents the actual hours worked by Jason}$$

Once the value of D is found, the value of J must be found.

For comparison, try using a ratio table to solve this problem. A third row is needed to keep track of the total hours worked:

| David	2	4	6	8	10	12	14	16
Jason	3	6	9	12	15	18	21	24
Total	5	10	15	20	25	30	**35**	40

The fifth way to solve proportion problems is by graphing. Consider the hamburger problem. The ratio table can be written vertically, making a table of values that can be graphed:

Pounds	People
3	12
6	24
9	36

A graph of these values is shown at right. A line can be drawn between the points because the x values between any of the data points represent possible amounts of hamburger. So, to answer the earlier problem, if we need to feed 30 people, we can read off the graph that we need 7.5 lbs of hamburger.

Connection between unit rate and slope: We can feed 12 people with 3 pounds of hamburger. This means that 4 people can be fed with 1 pound of hamburger. This is a unit rate, 4 people per pound. For every additional pound, we can feed 4 more people. In slope language, for every 4 people we go "up" on the graph, we go "over" 1 pound of hamburger. The slope is 4/1.

A general form of the proportion equation can be written like this: $\frac{4}{1} = \frac{y}{x}$ This can be read as 4 people are to 1 pound of hamburger as y people are to x pounds of hamburger. Multiplying both sides of the equation by x and switching the order of the quality, we have $y = \frac{4}{1}x$, which is in the form of the slope equation where the y intercept is 0 (i.e. that passes through the origin). **All proportional relationships pass through the origin.**

Hamburger recipe
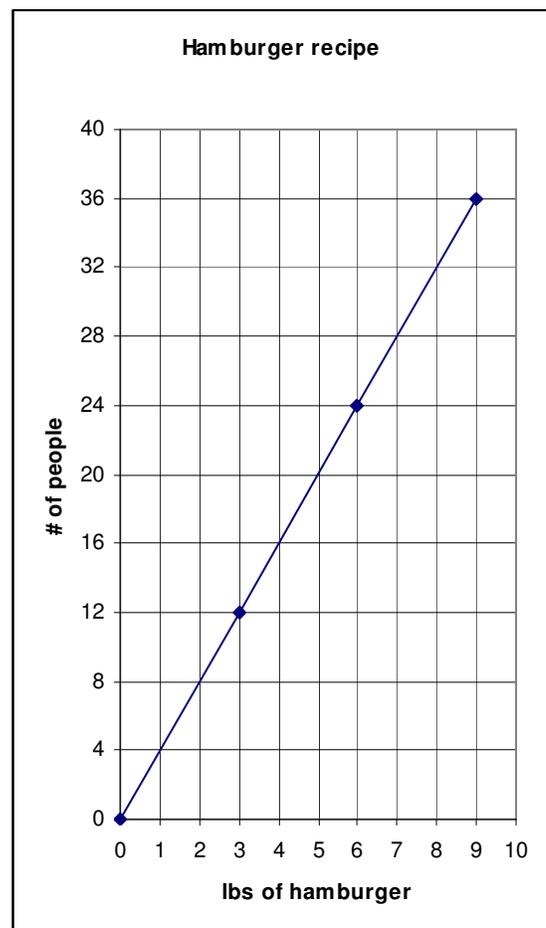
The point (0,0) is a solution to this proportion problem because 0 people can be fed with 0 pounds of hamburger.

When we set up this graph with "lbs of hamburger" on the x axis, we were designating that variable as the independent variable, the one that can fluctuate "on its own." That is, the question we are asking with this graph is, "If we have a certain amount of hamburger, how many people can we feed?" Of course, once the graph is drawn, it could be read to answer the other question: "If we have this many people, how many pounds of hamburger do we need?"

The unit rate, or slope in this equation, is the "constant of proportionality." Multiply number of pounds of hamburger (x) by 4 to get number of people (y).

The graph can be used to solve related proportion problems, such as *How many people can be fed with 7 pounds of hamburger?* The answer can be read from the graph as 28 people. *How many pounds of hamburger are needed to feed 35 people?* The answer can be approximated from the graph as 8.75.

> Note: The unit rate for the hamburger problem could have been calculated as ¼ pound per person. The choice of which value is the "unit" depends on the what the problem indicates is the independent variable. If the question is *Given x pounds of hamburger, how many people can we feed?* (that is, what is known is the amount of available hamburger), then the graph above is appropriate and the slope is 4 people per pound. If, instead, the question is *Given that we need to feed x people, how many pounds of hamburger would we need?* (that is, the number of people is known), the graph would be constructed with # of people on the horizontal axis and lbs of hamburger on the vertical axis. The unit rate (slope) would then be ¼ pound per person.

To summarize, the five ways to solve proportion problems are:

> **1) Find a unit rate.**
> **2) Use a ratio table.**
> **3) Use a "tape diagram."**
> **4) Set up and solve a proportion equation.**
> **5) Make a graph.**

Teaching sequence

The general teaching sequence for proportion problems is, first, to give a few simple proportion problems to students and let them solve them any way they can. Have them share their solutions. A variety of solutions should be chosen to "orchestrate" the discussion, allowing students to see connections across the solution approaches if possible.

Then each of the solution approaches should be practiced individually, with perhaps one day of examples and practice devoted to each approach, with some cumulative review and formative assessment each succeeding day. The emphasis should be on developing conceptual understanding first, then procedural fluency. As always, examples and practice problems should begin with numbers that work out fairly easily, to draw on students' number sense. Later problems can use numbers that result in fractional or decimal answers.

Break down the examples into a clear sequence. For the approach of writing proportion equations, have students practice translating word problems into the equation first. Make sure they write each ratio in the same order (e.g. red M&Ms to total M&Ms; time to distance; pounds of hamburger to people).

For the approach of finding a unit rate, allow students to choose which rate they want to use. In our examples, the two "hamburger" rates are 1/4 pound per person and 4 people per pound; for the driving example, students might say 50 miles per hour or 1/50 hour per mile (although they should be moved to the 50 miles per hour rate, since this rate is identified as speed). For the M&M problem, the unit rate is 8 M&Ms per 1 red M&M. It wouldn't make sense to say 1/8 red M&M for every M&M in the packet.

For the ratio table approach, introduce the practice of adding blank columns that aren't multiples of the given ratio, as with the driving situation. Students may also have to add columns "to the left" of the given ration, if the problem asks for a smaller amount. For example, a restaurant recipe calls for 10 cups of flour to make 4 cakes, but you want to make only 1 cake, a ratio table could look like this:

cups				10
cakes	1	2	3	4

The number of cups of flour needed for 2 cakes would be easy to compute (1/2 of 10), then the amount needed for 1 cake would be found by taking 1/2 again.

When working with proportion equations, always have students show each step of their work. For example:

$$\frac{150}{3} = \frac{400}{t}$$

$t \cdot \dfrac{150}{3} = \dfrac{400}{t} \cdot t$ Multiply both sides by t to get:

$t \cdot \dfrac{150}{3} = 400$

$t \cdot \dfrac{150}{3} \cdot 3 = 400 \cdot 3$ Multiply both sides by 3 to get:

$t \cdot 150 = 1200$

$t \cdot 150 \div 150 = 1200 \div 150$ Divide both sides by 150 to get:

$t = 80$

When working with graphs, use some problems where the graph is provided and students need to answer the problem by reading the graph. Also use some problems where students have to calculate the slope as a unit rate and relate it to the problem situation (this starts in 7th grade, where the connection between unit rate and steepness of the graph is introduced).