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# Modello numerico del pendolo in Excel

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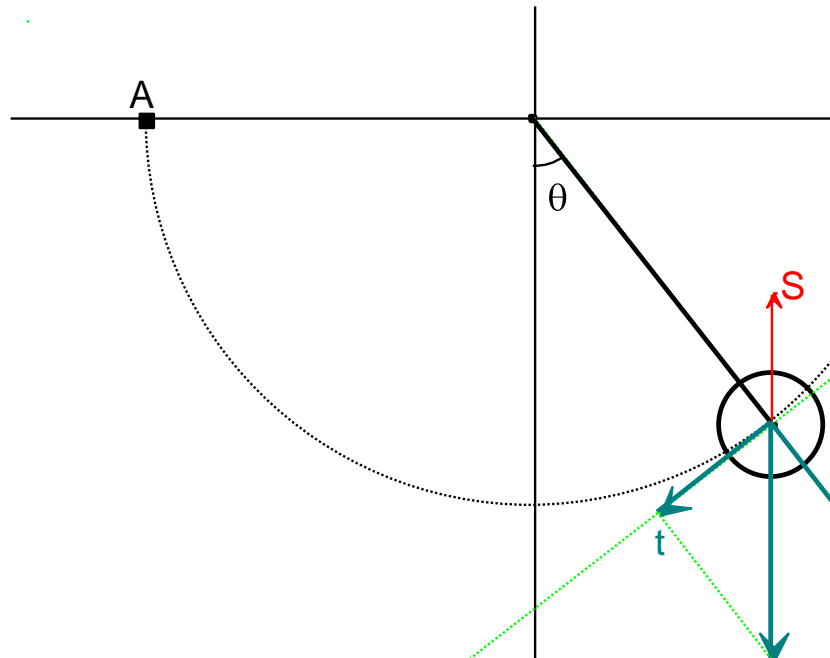
# Applicazioni

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- Studiare, variando densità e viscosità del mezzo, aspetti quali lo smorzamento delle oscillazioni o la relazione  $T-\vartheta$ .
- Verificare la legge fondamentale e ricercare il suo ambito di validità.
- Studiare la relazione  $T-g$ .
- Valutare la durata complessiva delle oscillazioni in funzione della lunghezza e della massa.

## Equazione del pendolo

$$-I_R \frac{d^2 \mathcal{G}}{dt^2} = M - M_a = \left[ (P - S)_{sfera} \cdot d_s + (P - S)_{asta} \cdot d_a \right] \sin \mathcal{G} - M_a$$



## Forze di attrito - 1

Attrito viscoso della sfera:

$$-M_a = \left[ C_{S1} \left( \frac{\pi \eta r d_s^2}{4} \right) \right] \omega + \left[ C_{S2} \left( \frac{\pi r^2 \rho d_s^3}{2} \right) \right] |\omega| \omega = g_{11} \omega + g_{21} |\omega| \omega$$

Attrito viscoso dell'asta:

$$-M_a = \frac{C_{F1}}{6} \eta (l^3 + l_1^3) \omega + C_{F2} \frac{s \rho (l^4 + l_1^4)}{8} |\omega| \omega = g_{12} \omega + g_{22} |\omega| \omega$$

$$-F_a = \underbrace{\frac{1}{2} C_A m_a a}_{\text{massa virtuale}} + \underbrace{6 C_H \pi r^2 \sqrt{\frac{\rho \eta}{2 \omega_0}} (\omega_0 v + a)}_{\text{forza di Basset}} \rightarrow$$

Massa virtuale e termine di Basset:

$$\begin{aligned} \rightarrow -M_a &= \left[ \frac{1}{2} C_A m_a d_s^2 + 6 C_H \pi r^2 \sqrt{\frac{\rho \eta}{2 \omega_0}} d_s^2 \right] \alpha + \left[ 6 C_H \pi r^2 \sqrt{\frac{\eta \rho}{2 \omega_0}} \omega_0 d_s^2 \right] \omega = \\ &= (C_A I_{V1} + C_H I_{V2}) \alpha + C_H g_{13} \omega \quad (g_{13} = I_{V2} \omega_0) \end{aligned}$$

Attrito nella sospensione:

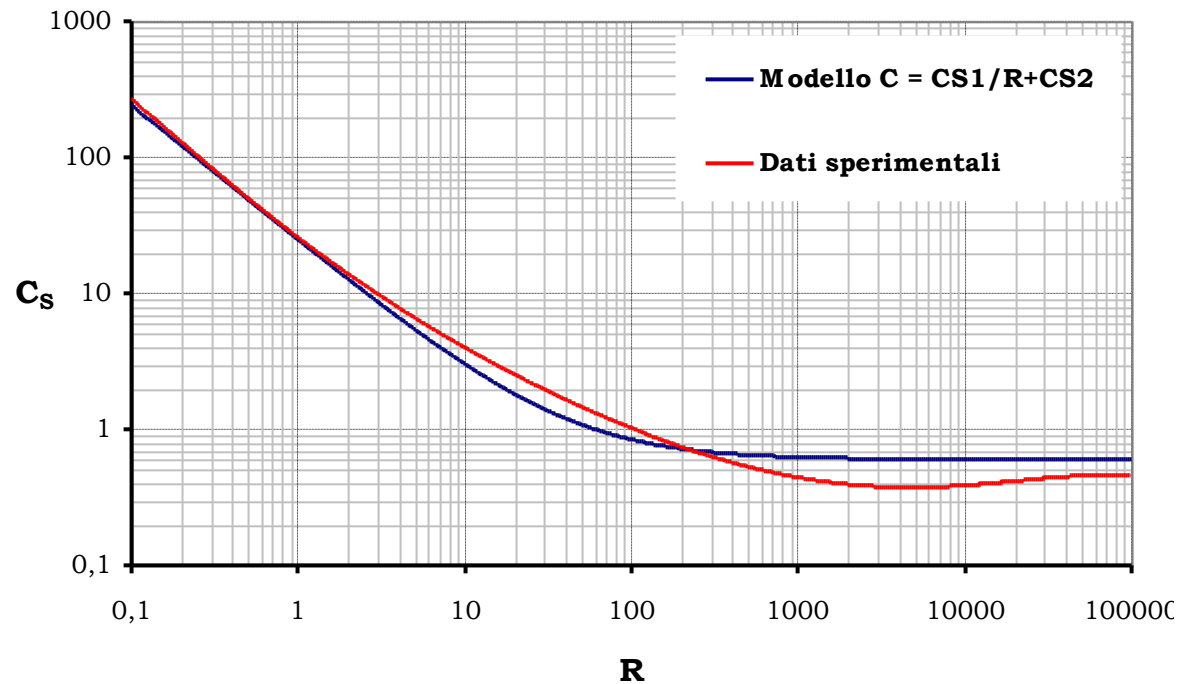
$$-M_a = k_0 \frac{|\omega|}{\omega} + k_1 \omega$$

## Attrito viscoso della sfera

$$-F_a = \frac{1}{2} C_S A \rho |v| v \quad C_S = f(R)$$

$$C_S = \frac{C_{S1}}{R} + C_{S2} = \frac{24}{R} + 0,6$$

$$R = \frac{2rp}{\eta} v = \frac{2rpd_s}{\eta} \omega$$



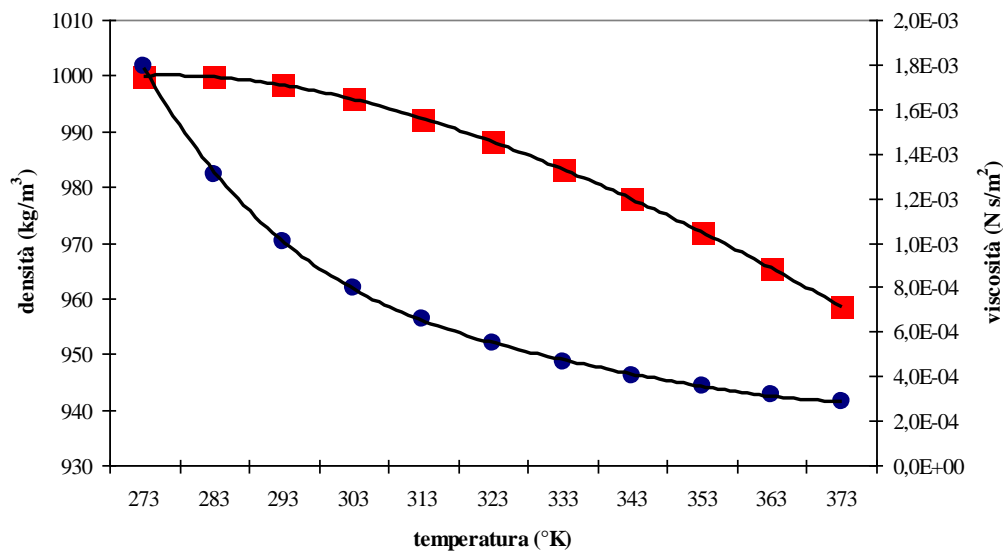
# Mezzo di oscillazione

Gas:  $\rho = \left(\frac{m}{k}\right) \frac{P}{T}$        $\eta = \begin{cases} \frac{(26,69)\sqrt{MT}}{\sigma^2\Omega(T^*)} & \text{se } l \ll \delta \\ 0 & \text{se } l \gg \delta \end{cases}$       (Chapman-Enskog)

## Densità e viscosità dell'acqua

■ densità       $y = -1,1364E-03x^4 + 4,3065E-02x^3 - 8,5702E-01x^2 + 2,0685E+00x + 9,9865E+02$   
 $R^2 = 9,9999E-01$

● viscosità       $y = 3,2261E-07x^4 - 1,0339E-05x^3 + 1,2781E-04x^2 - 7,8083E-04x + 2,4477E-03$   
 $R^2 = 9,9983E-01$



Liquidi:  
 dati tabulati a 20°C o 25°C

## Forma finale dell'equazione del pendolo

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$$-\frac{d^2 \mathcal{G}}{dt^2} = \frac{M}{I_{eff}} \sin \mathcal{G} + c_0 \frac{|\omega|}{\omega} + c_1 \omega + c_2 |\omega| \omega$$

$$\left\{ \begin{aligned} I_{eff} &= I_R + C_A I_{V1} + C_H I_{V2} \\ c_0 &= \frac{k_0}{I_{eff}} \\ c_1 &= \frac{g_{11} + g_{12} + C_H g_{13} + k_1}{I_{eff}} = \frac{g_1 + k_1}{I_{eff}} \\ c_2 &= \frac{g_{21} + g_{22}}{I_{eff}} = \frac{g_2}{I_{eff}} \end{aligned} \right.$$

I coefficienti  $c_0$ ,  $c_1$  e  $c_2$  vengono in genere determinati per confronto con dati sperimentali

Patrick T. Squire, *Pendulum damping*, Am. J. Phys. 54(11), November 1986

## Integrazione numerica: metodo Runge-Kutta 4° ordine

$$\begin{cases} \frac{\Delta \mathcal{G}}{\Delta t} = \omega \\ \frac{\Delta \omega}{\Delta t} = \alpha = f(\mathcal{G}, \omega, c_i) \end{cases}$$

$$\begin{cases} \theta_1 = \omega \Delta t & w_1 = -f(\mathcal{G}, \omega, c_i) \Delta t \\ \theta_2 = \left( \omega + \frac{w_1}{2} \right) \Delta t & w_2 = -f\left( \mathcal{G} + \frac{\theta_1}{2}, \omega + \frac{w_1}{2}, c_i \right) \Delta t \\ \theta_3 = \left( \omega + \frac{w_2}{2} \right) \Delta t & w_3 = -f\left( \mathcal{G} + \frac{\theta_2}{2}, \omega + \frac{w_2}{2}, c_i \right) \Delta t \\ \theta_4 = (\omega + w_3) \Delta t & w_4 = -f(\mathcal{G} + \theta_3, \omega + w_3, c_i) \Delta t \end{cases}$$

$$\begin{cases} \mathcal{G}_{fin} = \mathcal{G}_{iniz} + \frac{\theta_1 + 2\theta_2 + 2\theta_3 + \theta_4}{6} \\ \omega_{fin} = \omega_{iniz} + \frac{w_1 + 2w_2 + 2w_3 + w_4}{6} \end{cases}$$



# Il modello numerico in Microsoft Excel

The screenshot displays a Microsoft Excel spreadsheet titled "SimulatorePendolo". The interface is organized into several sections for configuring a pendulum simulation:

- Modello numerico del pendolo:** A yellow header section with an information icon.
- Environmental Parameters:** Includes fields for Latitude (0,6583731), height (0), Temperature (293), Pressure (1,01325000E+05), relative humidity (10%), and dimensions (1). A "Continua la simulazione" button is present.
- Buttons:** "Fourier", "Ricerca", "Grafici", and "Copia dati" buttons are arranged in a grid.
- Condizioni iniziali:** Initial conditions section with fields for  $t_0$  (0,00000000E+00),  $\Delta t$  (2,00000000E-02),  $\theta_0$  (1,57080E+00), and  $\omega_0$  (0,00000000E+00). A "1200,00" value is also visible.
- pendolo:** Field for "SIM1 (sfera: Ferro; asta: Nylon)".
- mezzo:** Field for "Aria".
- Attriti:** Friction parameters section divided into "Attrito viscoso" (C<sub>S1</sub>: 24, C<sub>S2</sub>: 0,6, C<sub>F1</sub>: 9, C<sub>F2</sub>: 1,2) and "Attrito nel perno" (k<sub>0</sub>: 3,60000000E-05, k<sub>1</sub>: 1,00000000E-06, k<sub>2</sub>: 0,00000000E+00).
- Parametri geometrici:** Geometric parameters section with fields for characteristic lengths (Z<sub>1</sub>: 0,00000E+00), arm volumes (V<sub>1</sub>: 0,00000000E+00), and sphere volume (V<sub>S</sub>: 4,18879020E-06).

The spreadsheet formula bar shows "teta\_0" with the value "=PI.GRECO()/2". The taskbar at the bottom shows the Start menu, taskbar with "Simulatore", "Microsoft PowerPoint - [...]", and "Microsoft Excel - Sim...", and a system tray with "IT" and "8.35".

### PublicSub Periodo\_Ampiezza()

Dim x(3), y(3)

On Error Resume Next

Sheets("calcoli").Range("B2:C60002").ClearContents

Sheets("calcoli").Range("F3:G60002").ClearContents

pos = 1

tempo = Sheets("dati").Cells(2, 1)

amp\_max = Abs(Sheets("dati").Cells(2, 2))

For i = 3 To 60002

If Abs(Sheets("dati").Cells(i, 2)) > amp\_max Then

tempo = Sheets("dati").Cells(i, 1)

amp\_max = Abs(Sheets("dati").Cells(i, 2))

punto = i

End If

If Sgn(Sheets("dati").Cells(i, 2)) <> 0 Then

If Sgn(Sheets("dati").Cells(i, 2)) <> Sgn(Sheets("dati").Cells(i - 1, 2)) Then

pos = pos + 1

Sheets("calcoli").Cells(pos, 2) = i

Sheets("calcoli").Cells(pos, 3) = Sheets("dati").Cells(i - 1, 1) - (Sheets("dati").Cells(i, 1) - Sheets("dati").Cells(i - 1, 1)) \* Sheets("dati").Cells(i - 1, 2) / (Sheets("dati").Cells(i, 2) - Sheets("dati").Cells(i - 1, 2))

If pos > 2 Then

For k = 1 To 3

x(k) = Sheets("dati").Cells(punto + k - 2, 1)

y(k) = Abs(Sheets("dati").Cells(punto + k - 2, 2))

Next k

$k = x(1)^2 * (x(2) - x(3)) + x(2)^2 * (x(3) - x(1)) + x(3)^2 * (x(1) - x(2))$

$a = (y(1) * (x(2) - x(3)) + y(2) * (x(3) - x(1)) + y(3) * (x(1) - x(2))) / k$

$b = (x(1)^2 * (y(2) - y(3)) + x(2)^2 * (y(3) - y(1)) + x(3)^2 * (y(1) - y(2))) / k$

$c = (x(1)^2 * (x(2) * y(3) - x(3) * y(2)) + x(2)^2 * (x(3) * y(1) - x(1) * y(3)) + x(3)^2 * (x(1) * y(2) - x(2) * y(1))) / k$

Sheets("calcoli").Cells(pos, 6) = -b / (2 \* a)

Sheets("calcoli").Cells(pos, 7) = (4 \* a \* c - b ^ 2) / (4 \* a)

Else

Sheets("calcoli").Cells(pos, 6) = tempo

Sheets("calcoli").Cells(pos, 7) = ampiezza\_max

End If

tempo = Sheets("dati").Cells(i, 1)

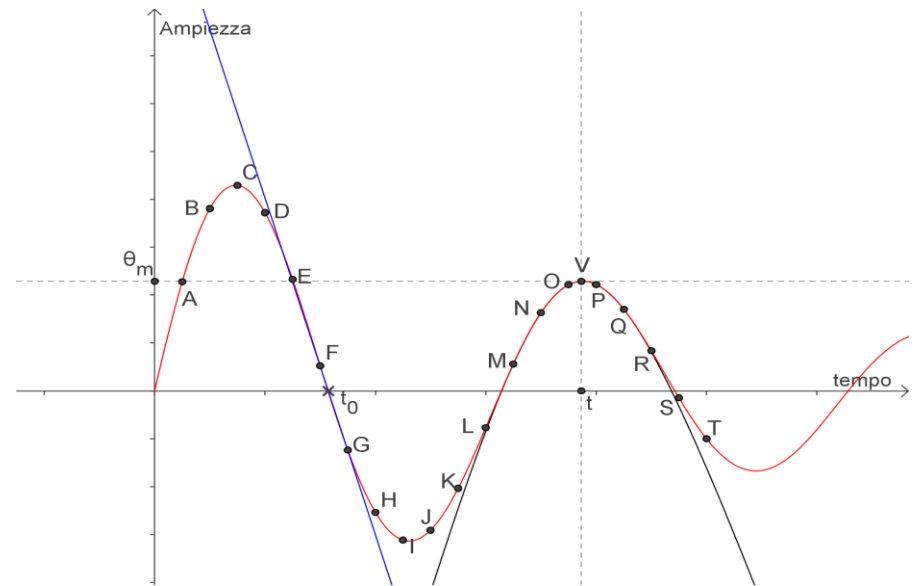
ampiezza\_max = Abs(Sheets("dati").Cells(i, 2))

End If

End If

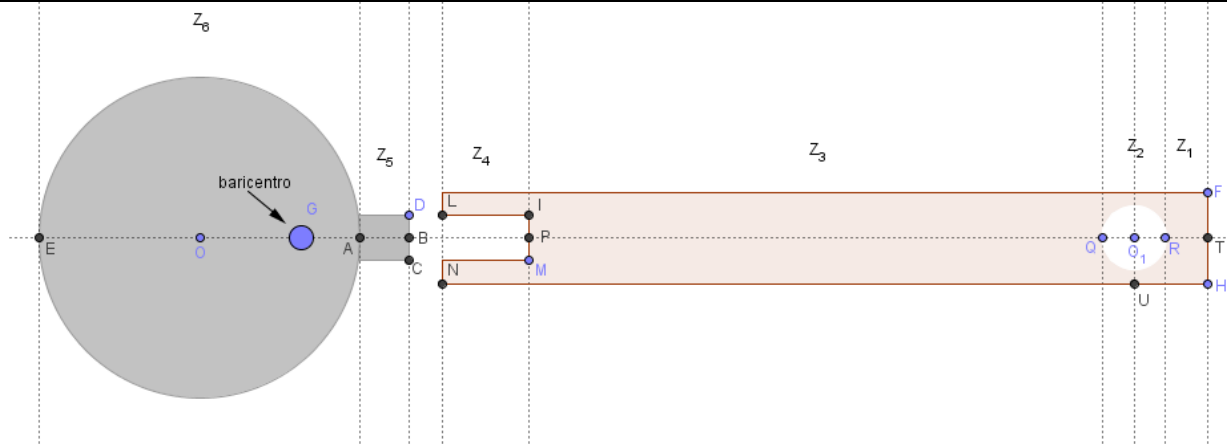
Next i

End Sub



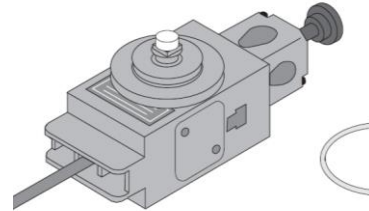
# Verifiche del modello

Identificativo	Asta o filo di sospensione						Sfera				
	$z_1$	$z_2$	$z_3$	$z_4$	$s$	$M_{asta}$	tipo	$z_5$	$s_5$	$r$	$M_{sfera}$
SA30 (solo asta 30 cm)	8,0E-03	4,0E-03	2,62E-01	2,3E-02	8,0E-03	1,143E-01	no	0,0E+00	0,0E+00	0,00E+00	0,000E+00
SA40 (solo asta 40 cm)	8,0E-03	4,0E-03	3,62E-01	2,3E-02	8,0E-03	1,536E-01	no	0,0E+00	0,0E+00	0,00E+00	0,000E+00
SA50 (solo asta 50 cm)	8,0E-03	4,0E-03	4,65E-01	2,3E-02	8,0E-03	1,940E-01	no	0,0E+00	0,0E+00	0,00E+00	0,000E+00
SC (asta + sfera cava)	2,3E-02	4,0E-03	3,69E-01	0,0E+00	7,3E-03	1,214E-01	cava	0,0E+00	0,0E+00	4,05E-02	1,016E-01
SNF1 (filo + sfera nera)	0,0E+00	0,0E+00	1,86E+00	0,0E+00	1,0E-03	1,665E-03	piena	0,0E+00	0,0E+00	1,75E-02	1,772E-01
SNF2 (filo + sfera nera)	0,0E+00	0,0E+00	1,24E+00	0,0E+00	1,0E-03	1,110E-03	piena	0,0E+00	0,0E+00	1,75E-02	1,772E-01
SP30 (asta 30 cm + sfera piena)	8,0E-03	4,0E-03	2,62E-01	2,6E-02	8,0E-03	1,143E-01	piena	9,0E-03	4,0E-03	2,00E-02	2,597E-01
SP40 (asta 40 cm + sfera piena)	8,0E-03	4,0E-03	3,62E-01	2,6E-02	8,0E-03	1,536E-01	piena	9,0E-03	4,0E-03	2,00E-02	2,597E-01
SP50 (asta 50 cm + sfera piena)	8,0E-03	4,0E-03	4,65E-01	2,3E-02	8,0E-03	1,940E-01	piena	9,0E-03	4,0E-03	2,00E-02	2,597E-01
SRF1 (filo + Sfera rossa)	0,0E+00	0,0E+00	1,85E+00	0,0E+00	1,0E-03	1,656E-03	piena	0,0E+00	0,0E+00	3,75E-02	1,192E-01
SRF2 (filo + Sfera rossa)	0,0E+00	0,0E+00	8,40E-01	0,0E+00	1,0E-03	7,521E-04	piena	0,0E+00	0,0E+00	3,75E-02	1,192E-01



# Apparato sperimentale

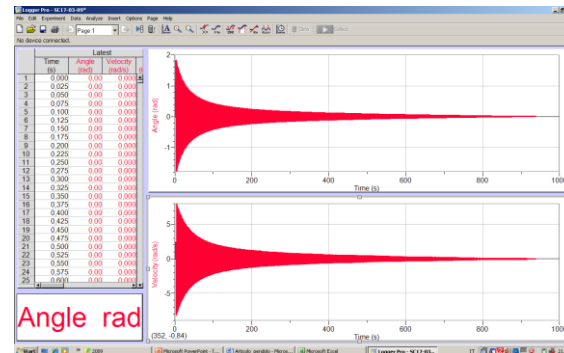
- Sensore di moto rotatorio CI-6625 (Vernier)



- LABPRO e LABQUEST (Vernier)

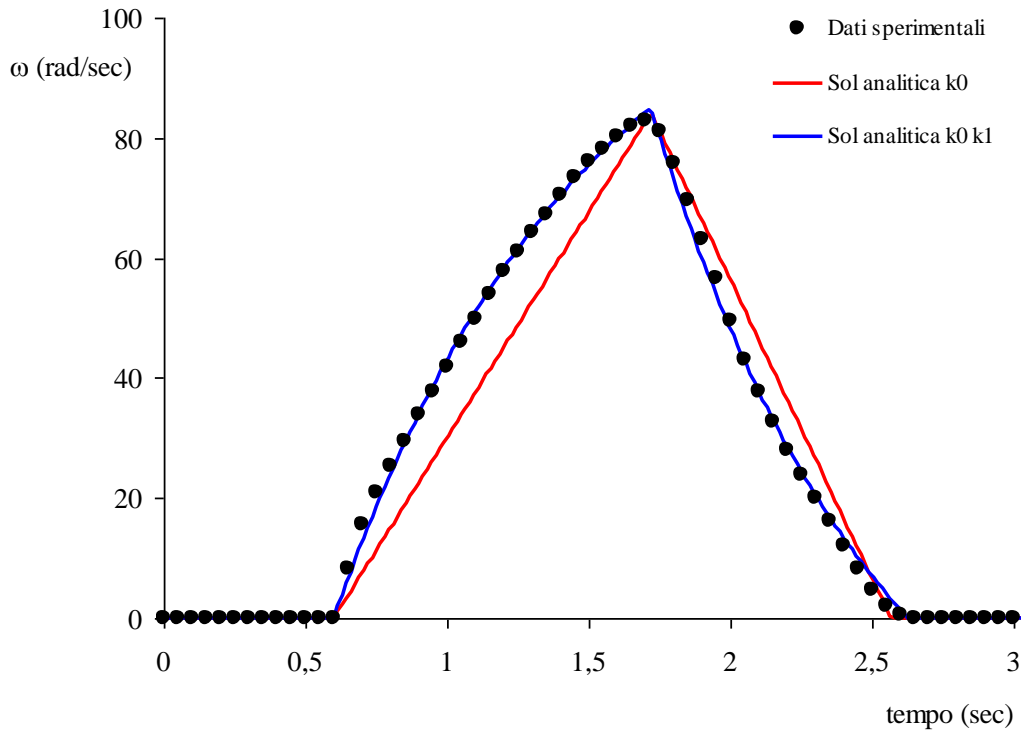


- Software Logger PRO



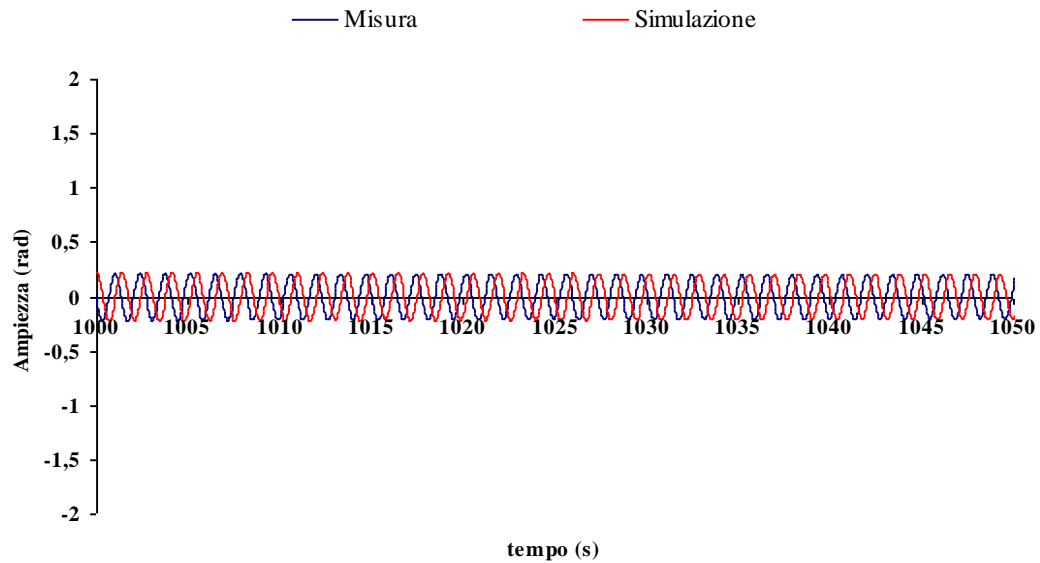
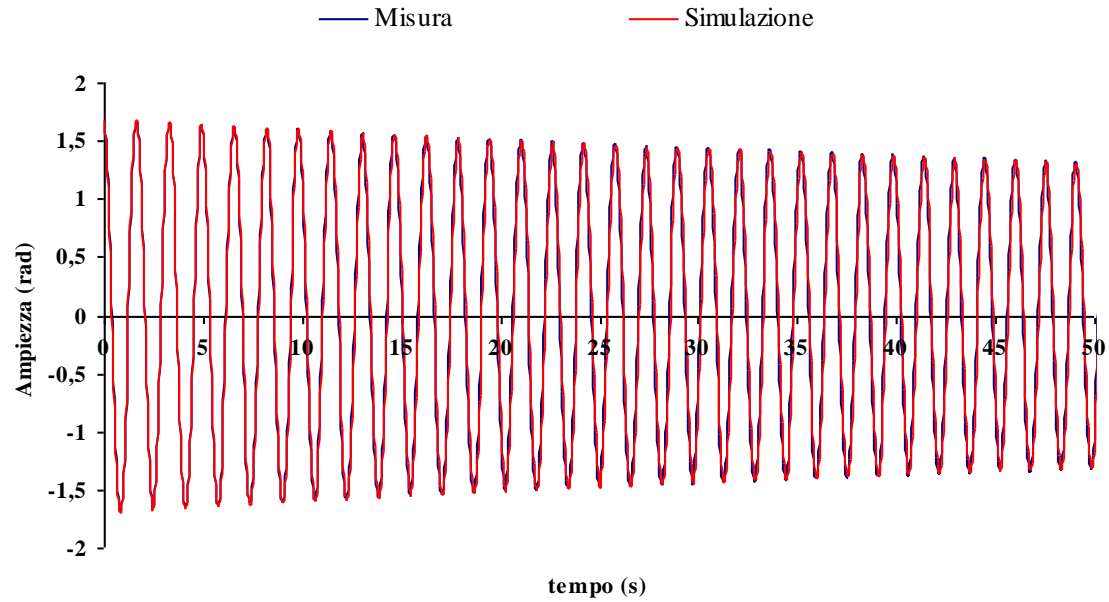
# Attrito nella sospensione

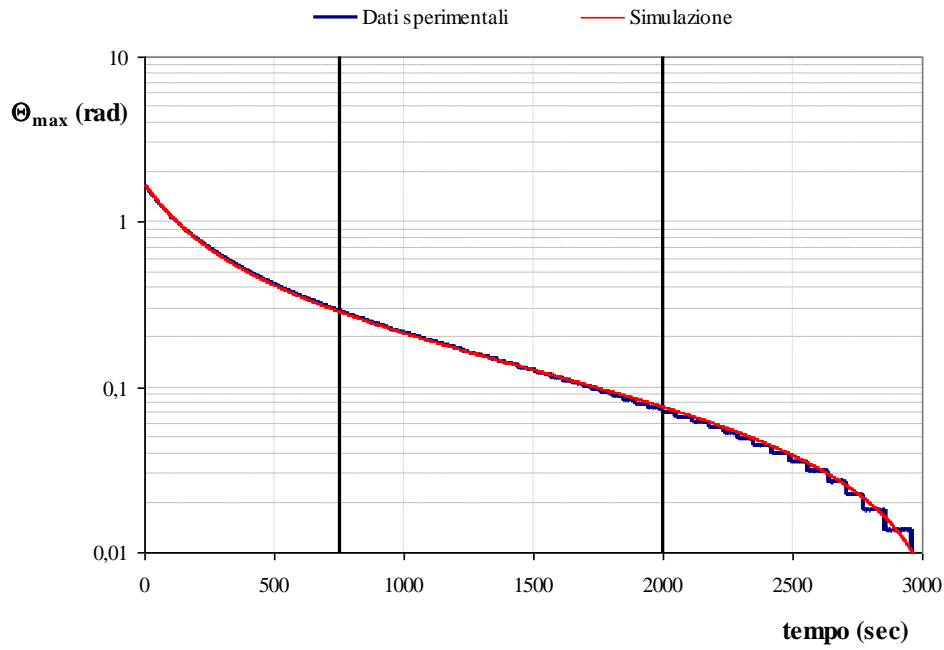
$$mgr - k_0 - k_1\omega = (I + mr^2) \frac{d\omega}{dt} \Rightarrow \begin{cases} \omega(t) = 0, & \text{per } t < t_0 \text{ e } t > t_2 \\ \omega(t) = \frac{mgr - k_0}{I + mr^2} \left[ 1 - e^{-\frac{k_1}{I+mr^2}(t-t_0)} \right], & \text{per } t \in [t_0; t_1] \\ \omega(t) = \frac{k_0 + k_1\omega_0}{k_1} e^{-\frac{k_1}{I}(t-t_1)} - \frac{k_0}{k_1}, & \text{per } t \in [t_1; t_2] \end{cases}$$



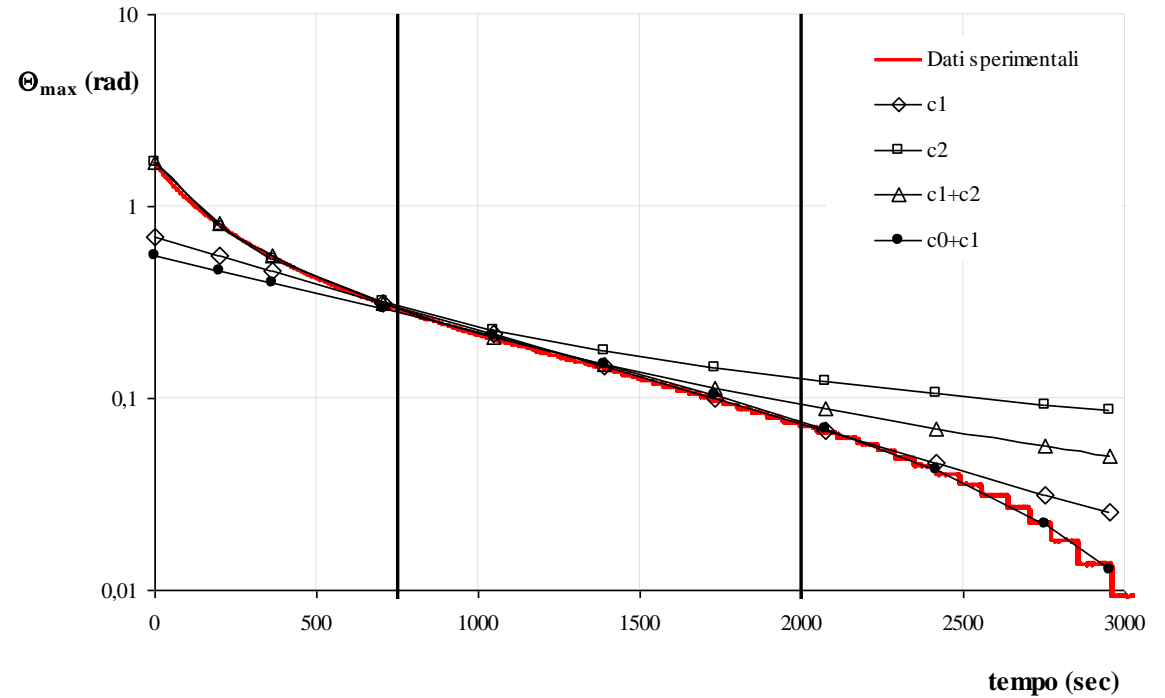
$$\begin{cases} I = (7,3 \pm 0,7) \cdot 10^{-7} \text{ kg} \cdot \text{m}^2 \\ k_0 = (3,6 \pm 0,9) \cdot 10^{-5} \text{ N} \cdot \text{m} \\ k_1 = (1,0 \pm 0,3) \cdot 10^{-6} \text{ N} \cdot \text{m} \cdot \text{s} \end{cases}$$

# Simulazione ed esperimento a confronto





## Diversi modelli di smorzamento a confronto



## International Gravity Formula + Free Air Correction

$$g_\phi = 9,780327 \cdot \left(1 + 5,3024 \cdot 10^{-3} \sin^2 \phi - 5,8 \cdot 10^{-6} \sin^2 2\phi\right) - 3,086 \cdot 10^{-6} h$$

$$\begin{cases} \phi = 0,65837 \text{ rad} \\ h = 0 \text{ m} \end{cases} \rightarrow g_\phi = 9,799687$$

pendolo semplice:  $T_0 = 2\pi \sqrt{\frac{l}{g}}$

pendolo fisico:  $T_0 = 2\pi \sqrt{\frac{I_R}{M}} = 2\pi \sqrt{\frac{I_R}{\left[ (m_s - m'_s) d_s + (m_a - m'_a) d_a \right] g}} = 2\pi \sqrt{\frac{I_R}{\Psi g}} \rightarrow g = \frac{4\pi^2 I_R}{\Psi T_0^2}$

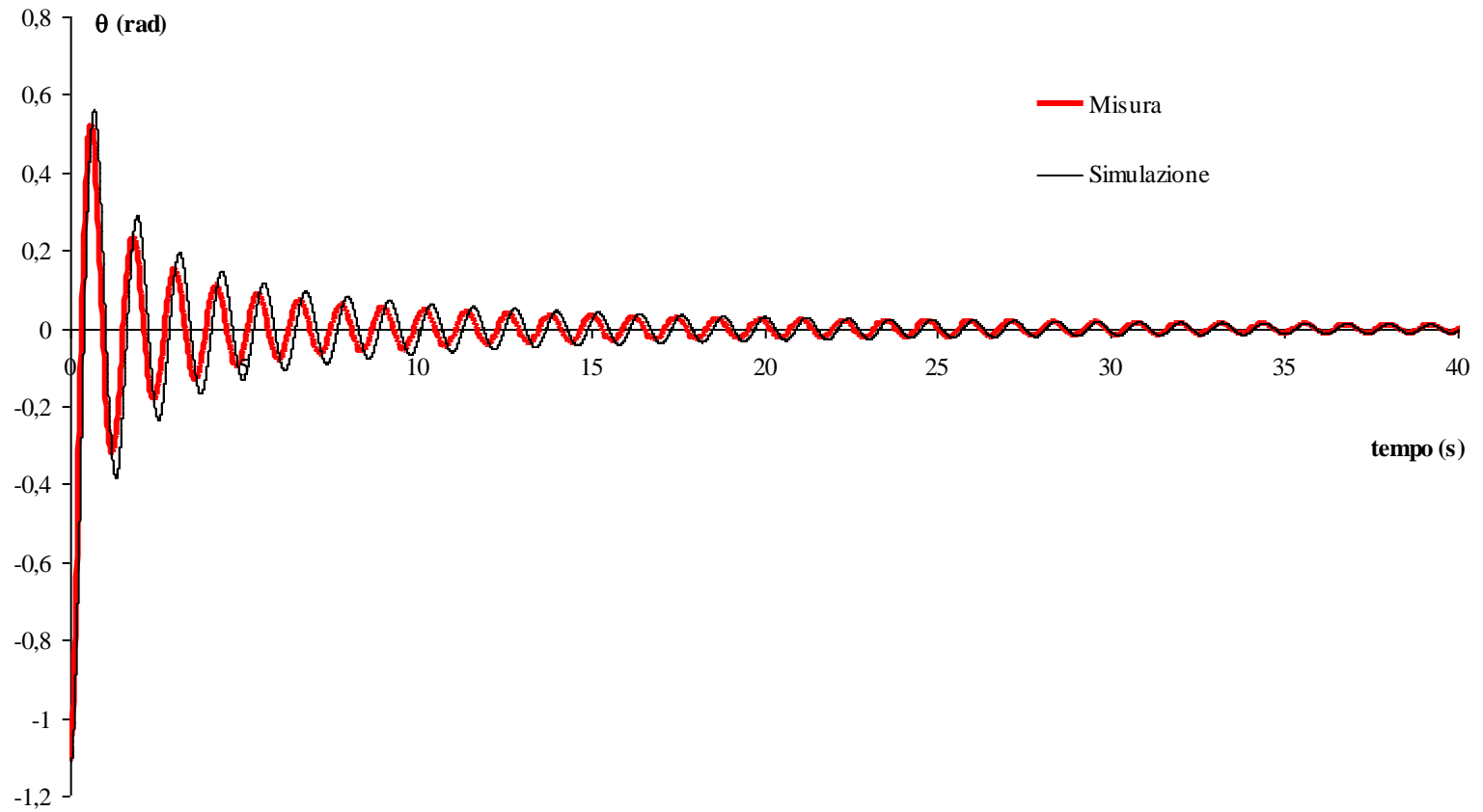
Pendolo	Periodo			
	Dati sperimentali	Modello numerico	S.A. pend. fisico	S.A. pend. semplice
SA30	0,871 ± 0,003	0,87105	0,87091	1,07526
SA40	1,013 ± 0,004	1,01311	1,01305	1,24862
SA50	1,141 ± 0,002	1,14129	1,14125	1,40499
SC	1,197 ± 0,004	1,19896	1,19796	1,28753
SNF1	2,745 ± 0,005	2,74686	2,74657	2,75020
SNF2	2,248 ± 0,005	2,24911	2,24888	2,25075
SP30	1,082 ± 0,003	1,08268	1,08260	1,11752
SP40	1,233 ± 0,004	1,23374	1,23364	1,28519
SP50	1,364 ± 0,005	1,36471	1,36460	1,43337
SP30 in acqua	1,21 ± 0,01	1,19861	1,16000	1,11752
SP40 in acqua	1,37 ± 0,01	1,36489	1,32179	1,28519
SRF1	2,756 ± 0,004	2,75747	2,75495	2,75751
SRF2	1,880 ± 0,002	1,88178	1,88024	1,88017

Calcolo di g		
ψ	I	g
0,01562	0,00294	9,79776
0,02866	0,00730	9,80070
0,04619	0,01493	9,80405
0,06361	0,02266	9,81546
0,33543	0,62811	9,81093
0,22403	0,28125	9,80738
0,09638	0,02804	9,81054
0,13549	0,05118	9,80989
0,17885	0,08267	9,80834
0,08395	0,03018	9,69279
0,11802	0,05506	9,81266
0,22730	0,42823	9,79225
0,10522	0,09234	9,80216

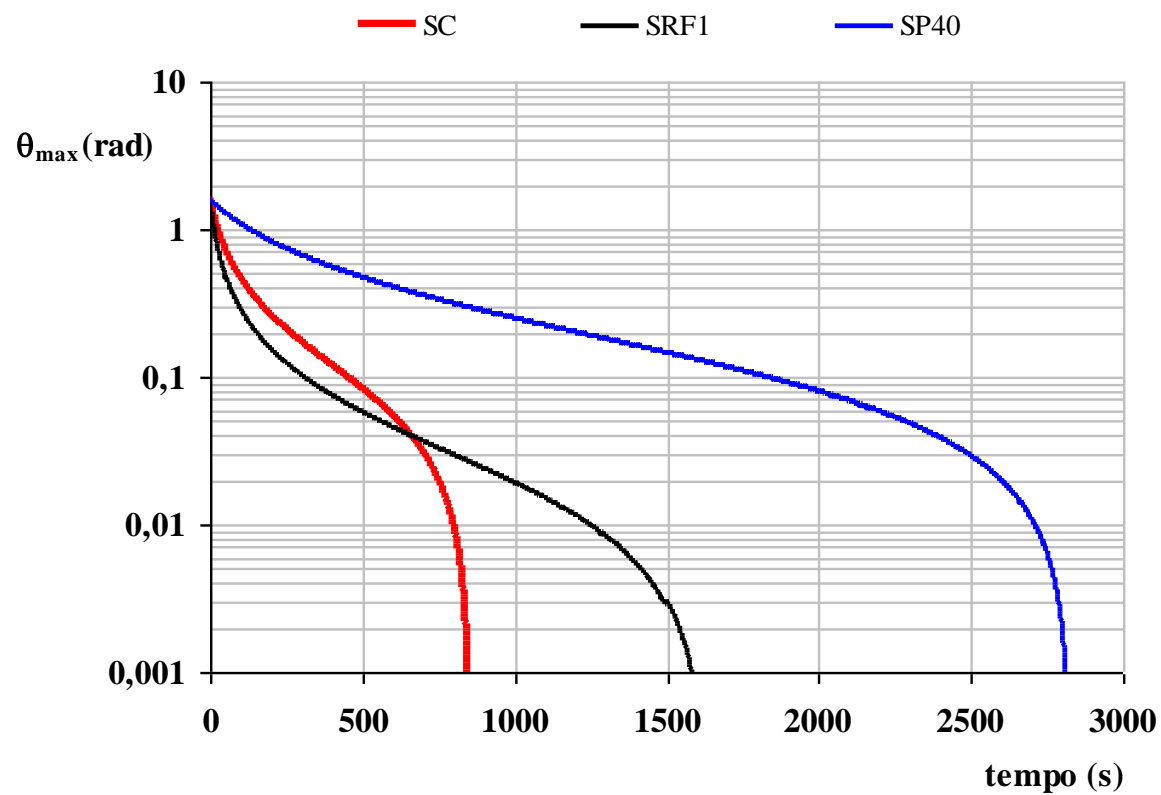
**Media** **9,805**  
**dev St** **0,007**



## Smorzamento delle oscillazioni in acqua



## Smorzamento delle oscillazioni per tre differenti pendoli



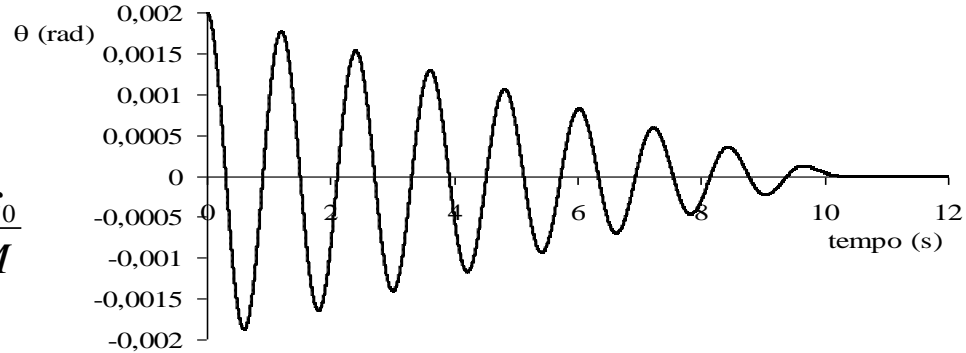
# Simulazioni: periodo delle oscillazioni

Grandi angoli: 
$$T = \frac{2T_0}{\pi} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} = T_0 \left( 1 + \frac{1}{4} g^2 + \frac{9}{64} g^4 + \frac{25}{256} g^6 + \frac{1225}{16384} g^8 + \dots \right)$$

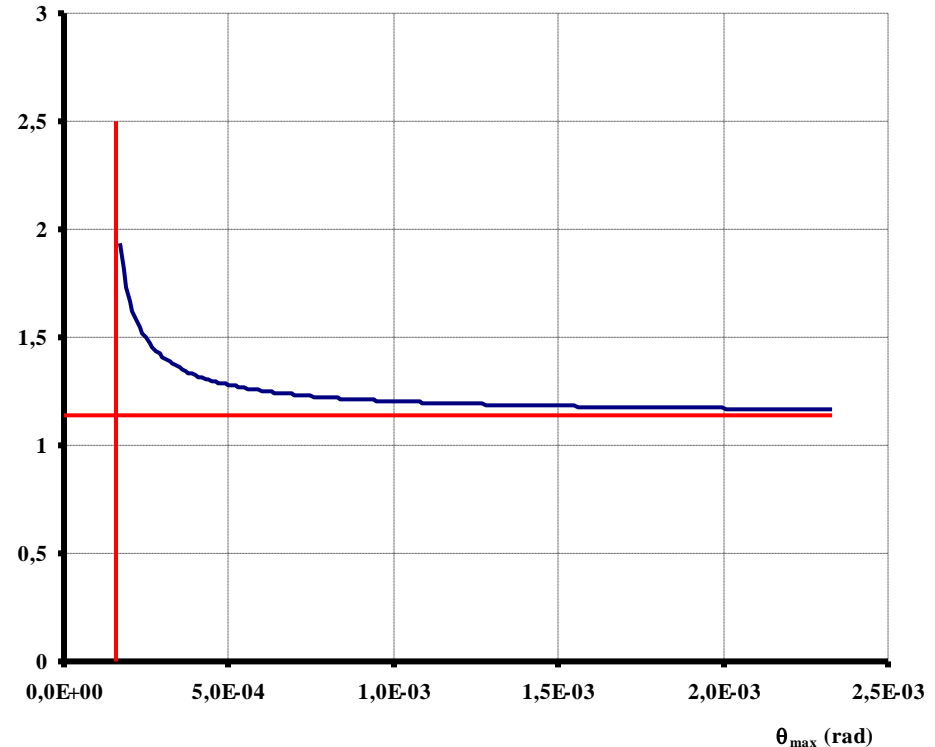
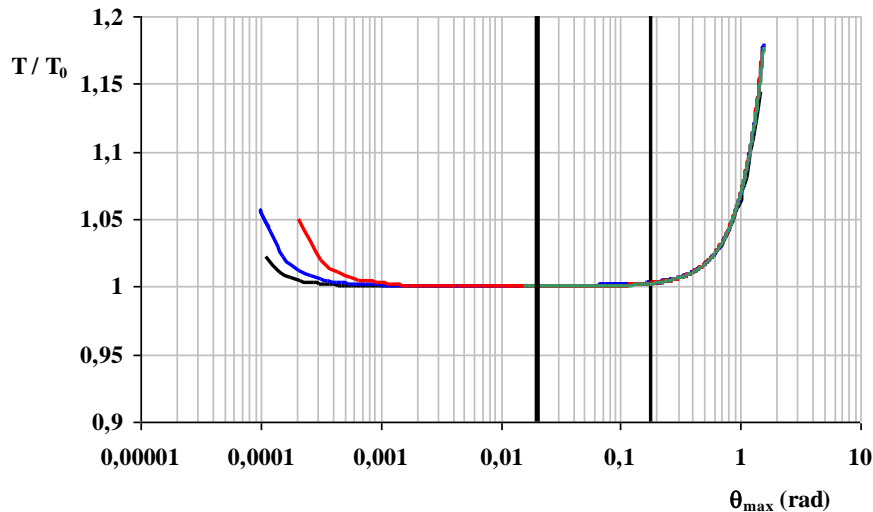
$$-\frac{d^2 \vartheta}{dt^2} = \frac{M}{I_{eff}} \vartheta + c_0 \frac{|\omega|}{\omega} = \frac{M}{I_{eff}} \vartheta - c_0$$

Piccoli angoli: 
$$\vartheta(t) = \left( \vartheta_{max} - \frac{Ic_0}{M} \right) \cos \sqrt{\frac{M}{I}} t + \frac{Ic_0}{M}$$

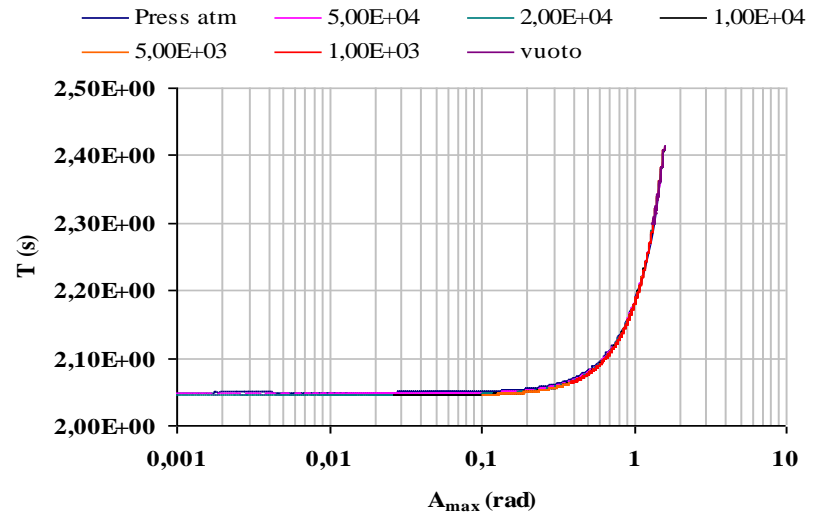
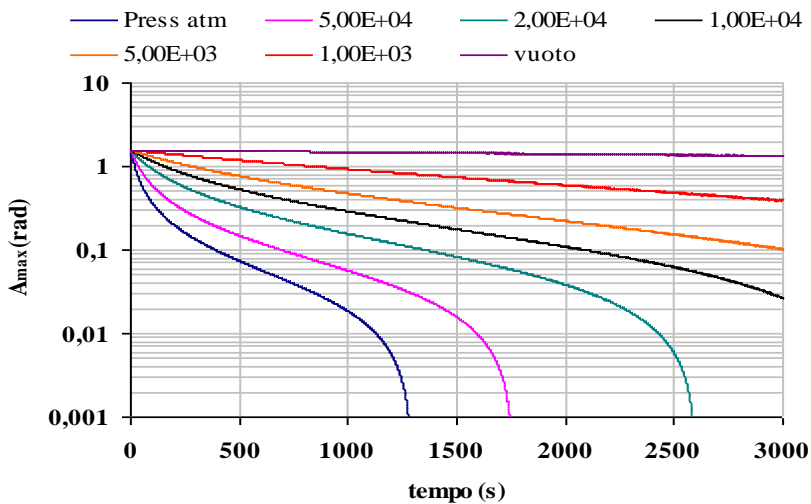
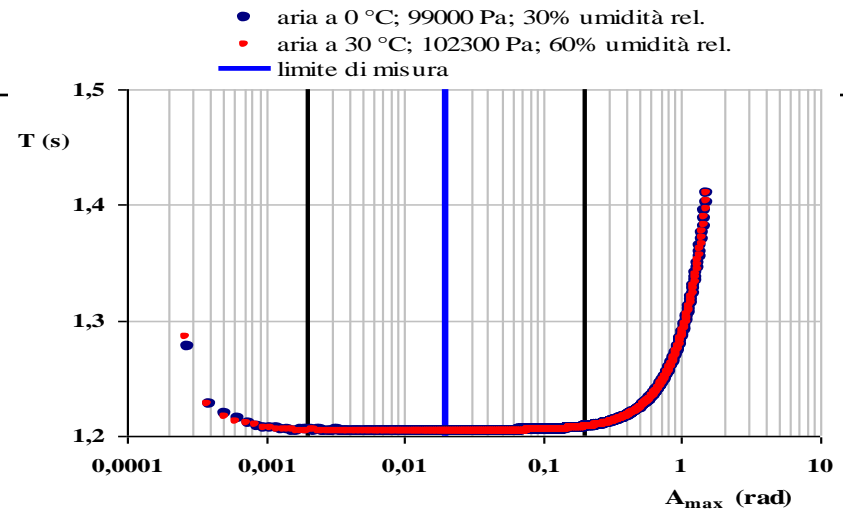
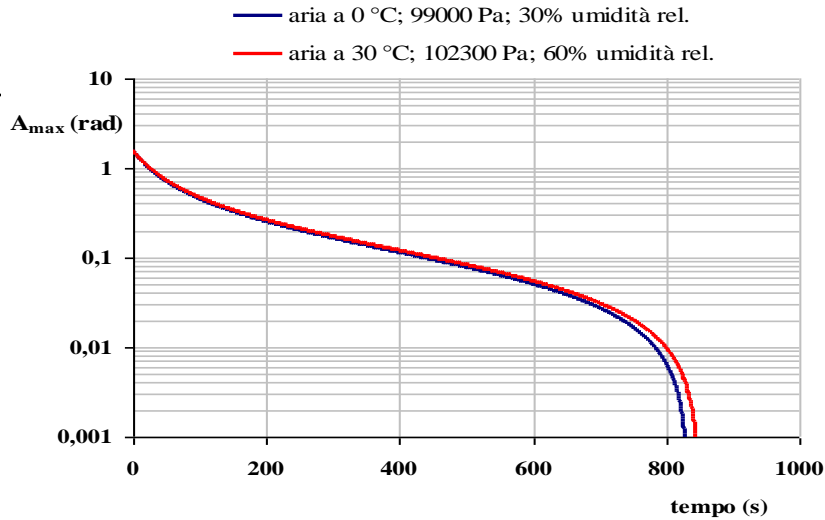
$$T = 4 \sqrt{\frac{I}{M}} \cos^{-1} \left( \frac{1}{1 - M \vartheta_{max} / Ic_0} \right) T(s)$$



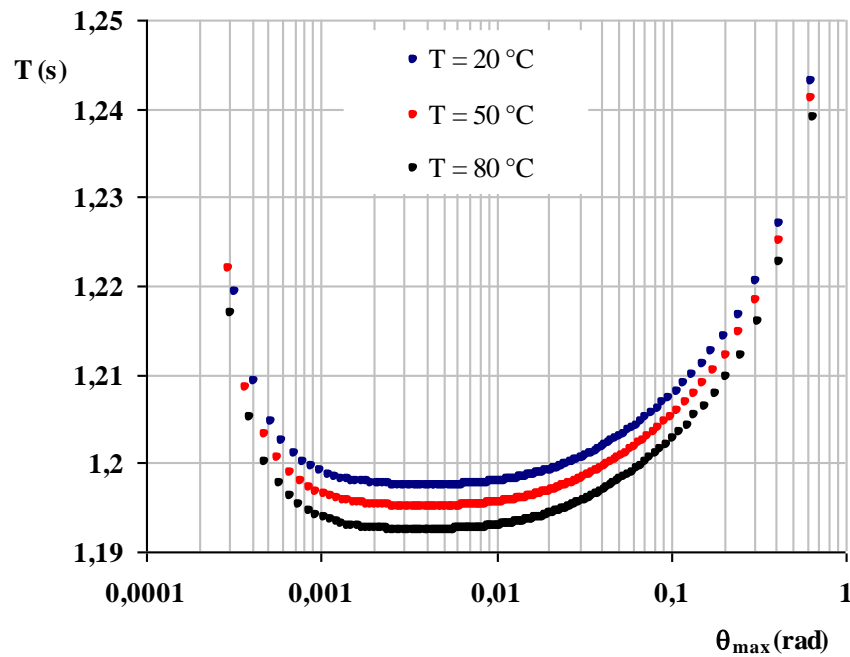
— SRF1    — SP40    — SC    — Integrale ellittico



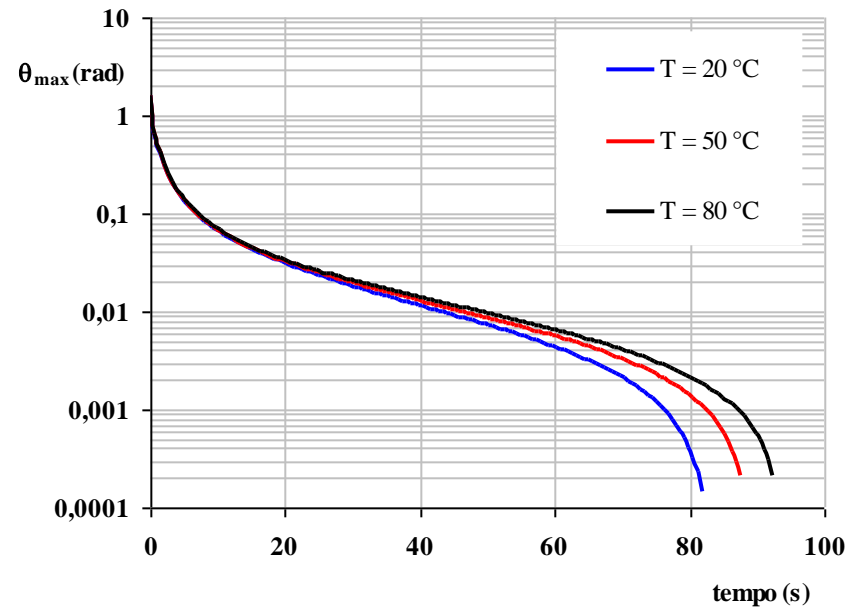
# Il pendolo in aria al variare di pressione e temperatura



# Pendolo in acqua

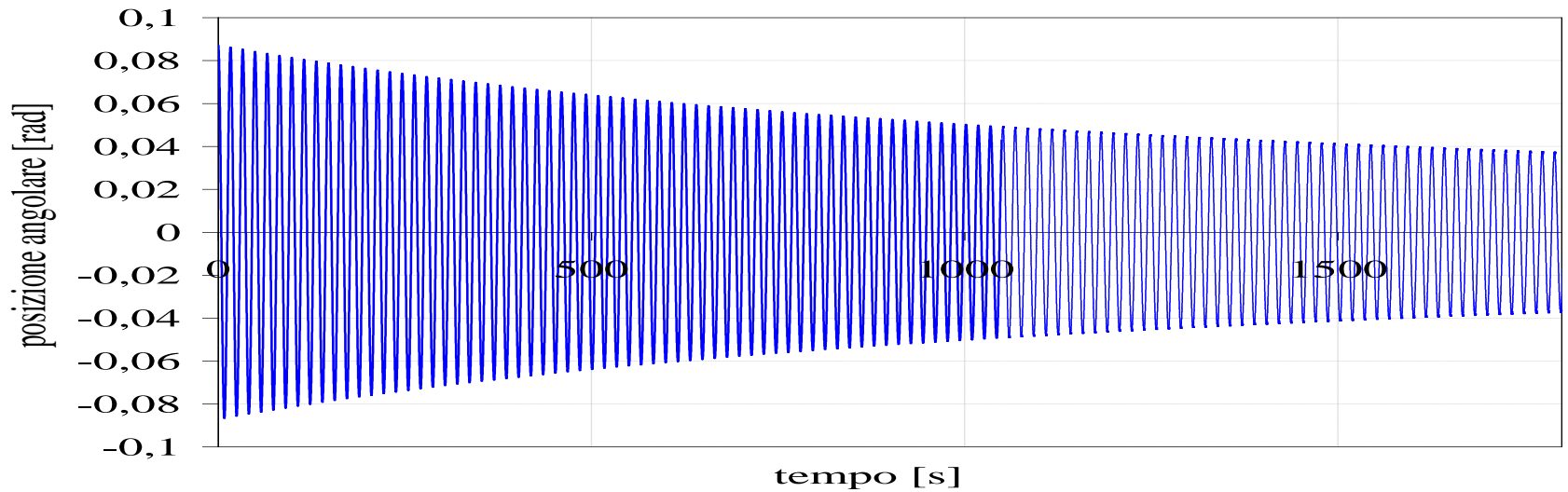
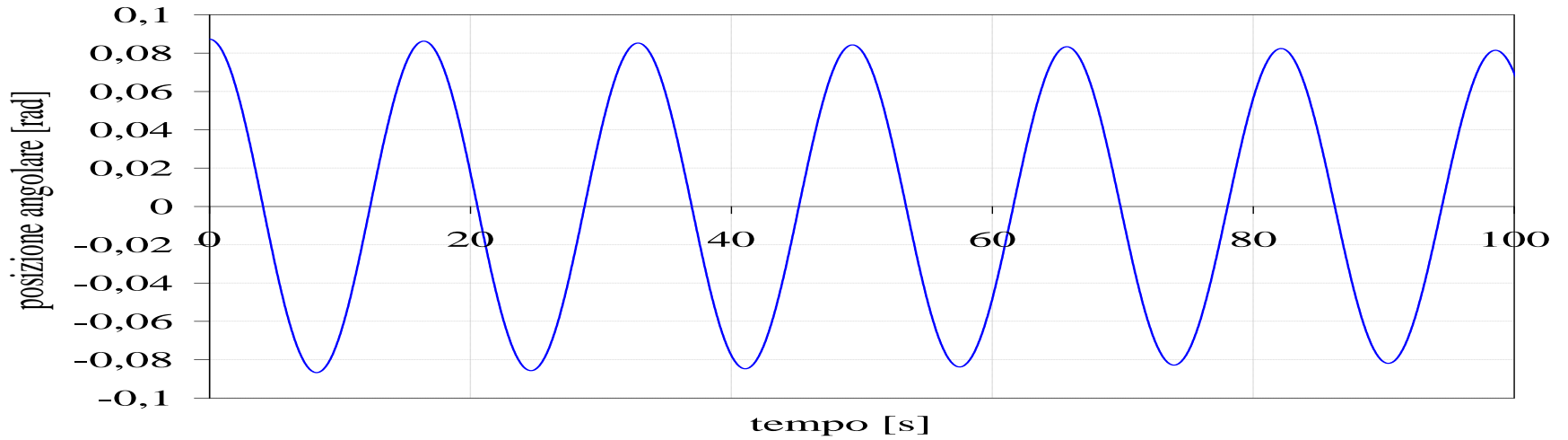


## SP30 in acqua

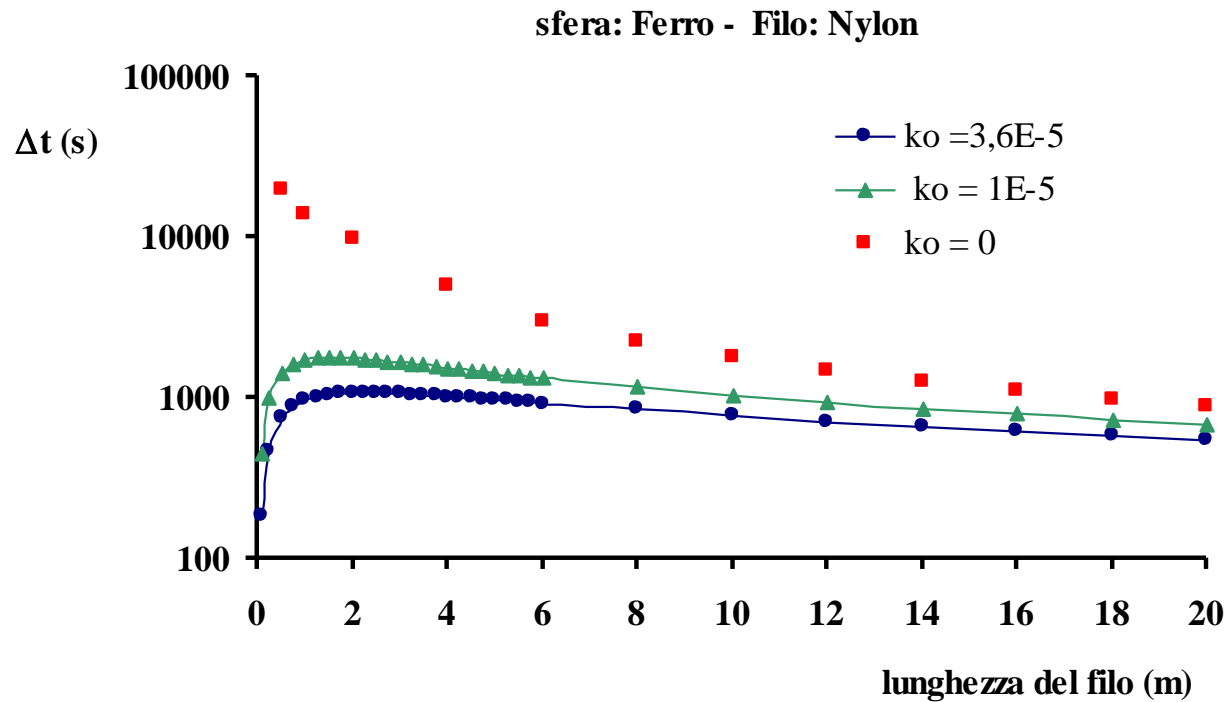


# Il pendolo di Foucault

$$l = 67 \text{ m} \quad d = 17 \text{ cm} \quad m = 28 \text{ kg} \quad \vartheta_0 = 5^\circ$$



# Durata complessiva delle oscillazioni



## Durata totale delle oscillazioni (da 0,1745 rad a 0,01 rad) vs. massa

