Teaching and Learning Calculus with Free Dynamic Mathematics Software GeoGebra

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Abstract

Research suggests that despite the numerous benefits of using technology in mathematics education, the process of embedding technology in classrooms is slow and complex (Cuban, Kirkpatrick, & Peck, 2001). GeoGebra is open-source software for mathematics teaching and learning that offers geometry, algebra and calculus features in a fully connected and easy-to-use software environment. It is available free of charge and used by thousands of students and teachers around the world in classrooms and at home. In this presentation we will both present applications of GeoGebra for calculus teaching at the high school and college level, as well as raise some of the implications of free and easy-to-use software such as GeoGebra for technology integration into the teaching and learning of calculus.

GeoGebra and Open Source

Computer algebra systems (such as Derive, Mathematica, Maple or MuPAD) and dynamic geometry software (such as Geometer's Sketchpad or Cabri Geometry) are powerful technological tools for teaching mathematics. Numerous research results suggest that these software packages can be used to encourage discovery and experimentation in classrooms and their visualization features can be effectively employed in teaching to generate conjectures (Lavicza 2006, Kreis 2004). However, different packages support teaching at a variety of curriculum levels and they require different amounts of classroom time for students to become proficient with the software. While computer algebra systems involve a considerable time commitment and their sophistication enables its use in upper level education, dynamic geometry software can be used as early as in elementary schools due to its mouse-driven user interface.

The multi-platform, open-source dynamic mathematics software GeoGebra (Hohenwarter & Preiner 2007) tries to combine the ease-of-use of dynamic geometry software with the versatile possibilities of computer algebra systems. The basic idea of the software is to join geometry, algebra, and calculus, which other packages treat separately, into a single easy-to-use package for learning and teaching mathematics from elementary through university level. GeoGebra is available free of charge on the Internet, has been translated to 36 languages by volunteers, and gathers a rapidly growing worldwide user community. Currently, the website www.geogebra.org attracts about 300,000 visitors per month from 192 countries, and it can be estimated that more than 100,000 educators use the software for their teaching around the world.

The open-source nature of the project has important implications both for educators and students. Unlike with commercial products, students are certainly not constrained to use the software only in schools or universities allowed by site licenses, but they can download and install GeoGebra on their private computers. For teachers, GeoGebra offers the powerful opportunity to create interactive online learning environments which has led many teachers to share free materials on the Internet. However, research suggests that, for the majority of teachers, solely providing technology is insufficient for the successful integration of technology into their teaching (Cuban, Kirkpatrick, & Peck, 2001; Ruthven & Hennessy 2002). It has been suggested that adequate training and collegial support boost teachers' willingness to integrate technology into their teaching and to develop successful technology-assisted teaching practices (Becker, Ravitz, & Wong, 1999). Today, GeoGebra users form a self-sustaining community that supports fellow users through an online user forum (www.geogebra.org/forum). In addition, teachers and researchers from all over the world are currently establishing an International GeoGebra Institute at various locations in order to contribute to the professional development of teachers, conduct research on GeoGebra, and continue to improve high quality software that is available free of charge for everyone (Hohenwarter & Lavicza 2007).

Calculus with GeoGebra

GeoGebra is being used for mathematics teaching from the primary to the university level. Teachers and researchers all over the world have developed numerous worksheets and methods using the software at many levels. Although teaching calculus concepts with GeoGebra is still an extensive area of development, the GeoGebraWiki (www.geogebra.org/wiki) already offers a wide ranging collection of calculus-related interactive worksheets. The following examples offer an overview of possible applications of GeoGebra for teaching basic concepts of calculus.

The following examples can be created with GeoGebra in a few minutes or even on the fly while teaching. They are fully dynamic, meaning that points can be moved along function graphs, parameters are changeable using sliders, and text adapts automatically to changes. In addition, all examples can be easily exported into dynamic web pages, so called *dynamic worksheets*, which contain an interactive applet with tasks for the students. After uploading these dynamic worksheets to the Internet, students are able to access them in school as well as at home without having to download or directly operate the GeoGebra software on their computers. By providing these different forms of dynamic and interactive figures, GeoGebra constructions can be integrated into mathematics classes in different ways (Little 2008).

Presentation – teacher centered approach: On the one hand, teachers can use a previously prepared GeoGebra file to present mathematical concepts to their students. By creating the corresponding dynamic figure prior to their lesson teachers save time during their classes and can use the construction for lesson planning. On the other hand, teachers could also create these dynamic figures on the fly during their lesson, allowing them a more flexible teaching style where they can react to students' questions, suggestions, and conjectures. Beginning users of technology tend to prefer the use of pre-made dynamic figures in their classroom, while more advanced GeoGebra users can use the software as a flexible teaching tool that is used to create mathematical models from scratch whenever they see fit.

Mathematical experiments – student centered approach: Depending on their computer skills and prior knowledge about the use of GeoGebra, students are able to explore and rediscover mathematical concepts in different ways using dynamic mathematics software. On the one hand, teachers can provide an (incomplete) interactive GeoGebra construction with accompanying questions and tasks on paper. These paper worksheets guide their students on an investigation of a certain mathematical concept, assuming that they already have some experience with operating the software itself.

On the other hand, teachers could also create self-contained dynamic worksheets prior to the lesson. With such pre-made worksheets, students don't need to operate the software GeoGebra itself but only work with an interactive html page in a browser. This saves time of teaching them how to operate the software itself. Being able to customize the user interface of the integrated interactive applets (e.g. showing or hiding the algebra window, reducing the number of available tools, displaying the toolbar help), teachers can decide beforehand how much freedom or guidance they want to provide for their students and which features and tools should be available for their students.

The construction files and dynamic worksheets described in the examples below give an overview of some basic calculus concepts that can be visualized and investigated using GeoGebra. Such dynamic visualizations can support mathematical experiments, connections between symbolic and graphical representations, and discussions about conjectures and basic concepts.

Example 1: Secant and Tangent Line of a Function

Our first example can help with visualizing and supporting the understanding of the difference quotient, to grasp the connection between secant and tangent lines, as well as to understand the importance of the limit concept and the differential quotient in this context. Figure 1 shows the graph of a function *f(x)* as well as two moveable points $A=(a, f(a))$ and $B=(b, f(b))$ that lie on the function graph. The specific values of the difference quotient $\frac{f(b)-f(a)}{b-a}$ are shown as dynamic text that adapts when *A* or *B* are moved with the mouse. Furthermore, the secant line through points *A* and *B* as well as its slope are shown.

This dynamic figure can be used by students to investigate the difference quotient as a numerical approximation of the slope of the tangent line by dragging point *B* along the graph of *f(x)* towards point *A*. They experience that when both points merge to become one, the secant line disappears and the difference quotient becomes undefined. This "problem" is a good starting point for interesting discussions in the classroom: Why does the secant line disappear? How can we try to solve this situation to get a value for the slope? In order to visualize the special case of the secant 'transforming' into the tangent, the tangent line can be displayed using the checkbox shown in the upper right corner of the construction. Instead of giving students an answer to a problem they didn't have in the first place, such explorations allow a more meaningful introduction of the abstract concept of the differential quotient as a solution to a problem students experienced themselves.

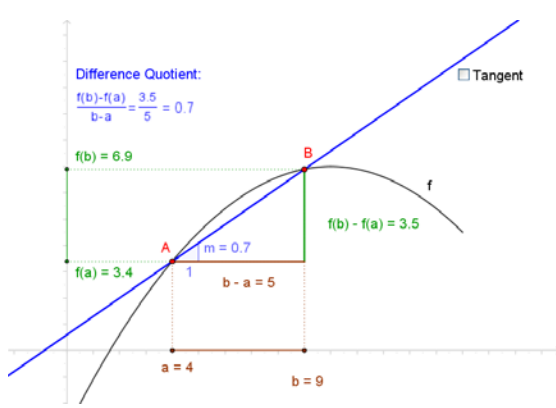

Fig. 1: Secant and tangent line of a function

Example 2: Tracing the Slope Function of *sin(x)*

The concept of derivative also potentially creates difficulties among students who lack proper understanding and visualization skills. To motivate the transition from the 'tangent-slope-problem' towards the derivative of *f(x)* we could experimentally introduce the graph of the *slope function* (see Fig. 2). In this example, the tangent to *f(x)* through a moveable point *A* that lies on the function graph and the tangent's slope are used to create a new point *m*. By moving point *A* along the function graph the trace of point *m* draws the corresponding slope function, i.e. the derivative, of the initial function *f(x)*. After coming up with a conjecture about the equation for this slope function, students can check their assumption by typing the equation into GeoGebra's input field and comparing the corresponding function graph with the experimentally created trace of the slope function.

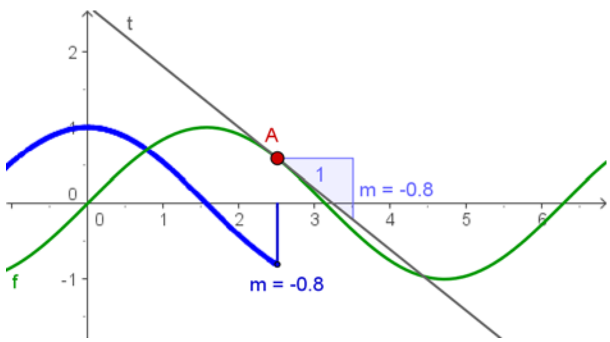

Fig. 2: Tracing the slope function of sin(x)

Since GeoGebra allows students to modify the initial function *f(x)* any time, this dynamic figure can also be used to investigate all standard functions and their derivatives. Additionally, it can help students to explore different rules for derivations, for example: Why does a constant term within the initial function's equation have no impact on the derivative? By using a worksheet with guiding questions, students can systematically analyze the derivatives of different polynomial functions (e.g. $f(x) = x + 2$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = 4x^2$). After writing down their findings in form of a table they might be able to spot a pattern and discover some rules of derivation by themselves.

Example 3: Derivatives, Roots, Extremal Points

Figure 3 visualizes a polynomial $f(x) = ax^3 + bx^2 + cx + d$, its root(s) *R*, extremal point(s) *E*, point of inflection *I*, and derivatives *f'* and *f''*. The parameters of polynomial function *f(x)* can be modified using sliders allowing students …

(a) to investigate the parameters' influence on the initial function (e.g. parameter *d* moves the function graph 'up' and 'down' without changing its shape).

(b) to explore the connections between the initial function and its derivatives (e.g. parameter *d* doesn't influence the derivatives at all; where *f* has an extremal point the first derivative f' has a root).

A systematic analysis of such characteristics of polynomials and their derivatives could possibly help students to better understand their algebraic manipulations of functions, to visualize characteristics of certain types of functions, and to improve their skills of sketching graphs of functions and their derivatives.

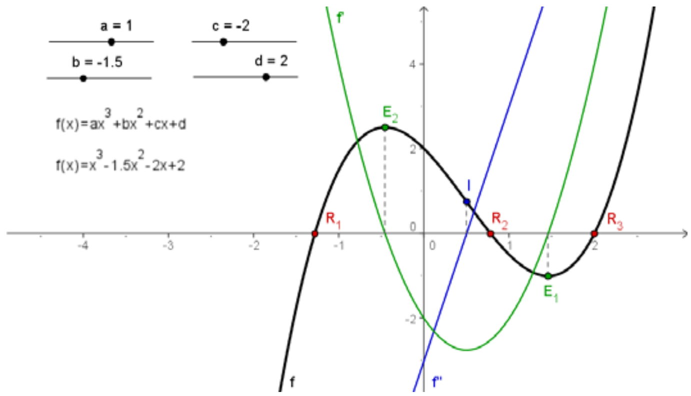

Fig. 3: Derivatives, roots, extremal points

Example 4: Dynamic Taylor Polynomial

In this example, we go beyond the usual scope of high school calculus by having a dynamic look at Taylor polynomials. Figure 4 shows the graph of the cosine function as well as the point of approximation *A* which can be moved along the *x*-axis in GeoGebra. Using a slider for *n*, students are able to change the number of terms of the Taylor polynomial and thus, influence the grade of accuracy of the approximation. Since point *A* can be dragged with the mouse, the initial function can be approximated at different places. At the same time, the dynamic text displaying the equation of the corresponding Taylor polynomial automatically adapts to all modifications of the dynamic figure.

In addition, the initial function *f(x)* can be modified at any time, allowing the investigation of Taylor polynomials for a variety of functions. The integration of multiple representations into everyday calculus classes (graph – dynamic visualization, equation – algebraic representation) can help students to better understand the concept of Taylor approximations as well as the meaning of the point of approximation and grade of accuracy.

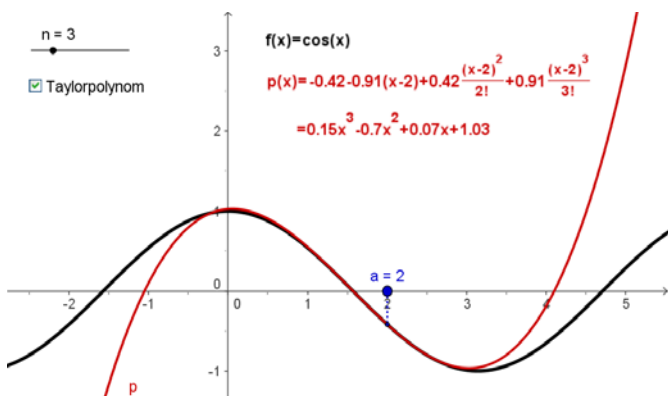

Fig. 4: Dynamic Taylor polynomial

Example 5: Upper and Lower Sums

Figure 5 dynamically visualizes the concept of the Riemann integral using lower and upper sums. In addition to the graph of function *f*, two points *a* and *b* are shown that can be moved along the *x*-axis in order to modify the investigated interval. Using a slider *n*, students can change the number of rectangles used to calculate the (equidistant) lower and upper sum in respect to function *f* within this interval. Both the upper and lower sum values as well as their difference are displayed as dynamic text which automatically adapts to modifications. Students can show or hide different components of the construction using checkboxes in order to keep focused on a certain task. Using this dynamic figure, students can explore different aspects of the Riemann integral guided by questions and tasks like the following:

1. Use slider *n* to find out how the rectangles for the lower sum / upper sum are constructed.
   a. Express the width of a rectangle in terms of the interval length $b-a$ and the number of rectangles *n*.
   b. Describe how the height of a certain rectangle can be found for the lower resp. upper sum.
2. Describe what happens to the values of the lower and upper sum when you increase the number of rectangles?
3. Imagine slider *n* being infinitely long, allowing you to create an infinite number of rectangles for the calculation of the lower sum / upper sum.
   a. What happens to the width of a single rectangle when *n* goes to infinity?
   b. What would happen with the values for the lower sum and upper sum when you move *n* to infinity?

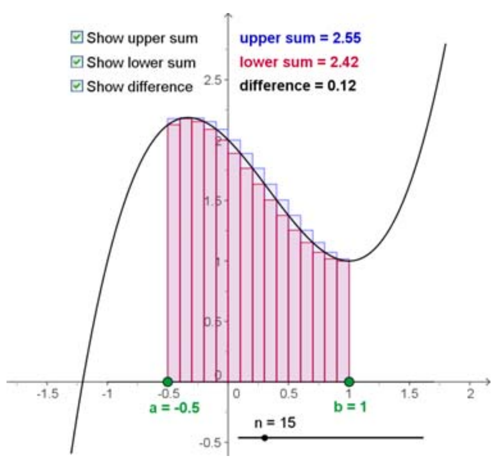

Fig. 5: Upper and lower sums

Example 6: Tracing the Area Function

The last example presented here is similar to tracing the slope of a function (see example 2). This time, we are investigating the *area function* as way to connect the idea of the definite integral with anti-derivatives. Two moveable points *A* and *B* on the graph of a function *f* determine an interval which delimits the area between the function graph and the *x*-axis of the coordinate system (see Fig. 6). This area value is used to create an additional point *area* that follows the right interval limit and creates the trace of the area function.

Again, the initial function *f* can be modified at any time allowing students to experimentally find the area function for a variety of function graphs before anti-derivatives are discussed. Similar to example 2, students could again use this dynamic figure to find the equation of the area function for a variety of different polynomials and check their conjectures using GeoGebra. Being guided through a systematic analysis of several types of functions students might be able to rediscover rules of integration by themselves.

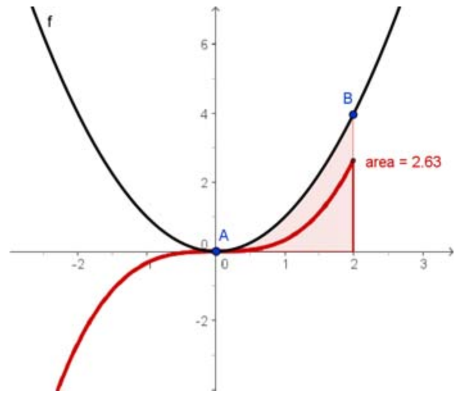

Fig. 6: Tracing the area function

Experiences with using GeoGebra in Calculus Classes

In 2006, similar dynamic GeoGebra constructions and interactive figures have been combined into interactive learning environments in a project for Austrian high schools. Supported by their teachers and by using different kinds of instructional materials (e.g. 'traditional' worksheets on paper, interactive applets, quizzes) students were guided towards discovering the concepts of derivative and / or integral. These learning environments were tested in Austrian high schools with several hundred students (Embacher 2006). Overall, participating students found the dynamic and interactive material helpful to understand and visualize underlying mathematical concepts. The following three statements by students after working with interactive GeoGebra calculus material for one week have been characteristic:

> "It is helpful when you see what is changing when you change something else."

> "If you move point *B* towards point *A*, you realize that the secant becomes the tangent line. With a drawing on paper I never was able to visualize what this would look like."

> "You can experiment a lot and try out a lot of things to find your own solution to problems."

While GeoGebra is widely used in middle and high schools, especially in Europe, its use at university level is also emerging. There are several educators in the USA who have published interactive university-level calculus material on the Internet. They have reported that creating these materials in GeoGebra was easier and less time consuming than with other software (Hohenwarter, Taeil, & Preiner 2007), for instance:

> "Don't know how to do it in Maple, but I just put up an applet that does this. On Monday, I saw a presentation […] of GeoGebra, and it took me about 20 minutes to figure out how to do polar coords in this program and create the page:
> http://www.sonoma.edu/users/f/fordb/polarcoordstrace.html."
> (Ben Ford, Sonoma State University, California)

They also stated that such interactive constructions have the potential to facilitate the teaching of certain calculus concepts and that students can benefit from the integration of dynamic visualizations into their 'traditional' calculus classes:

> "One of the features in GeoGebra is 'Navigation bar for construction steps' which is an important tool to export constructions as interactive web pages. […] Using 'Navigation bar' teachers can give better and accurate drawings and explanations. Even after the drawing and/or construction the teacher is able to change the shape of the drawing by changing parameters or moving points. This gives a much better insight and understanding of the subject to the students."
> (Taeil Yi, University of Texas at Brownsville)

Conclusion

In this paper, we highlighted various opportunities that dynamic mathematics software like GeoGebra can offer for the teaching of calculus. We have provided examples on how GeoGebra can be used in classrooms to explore basic calculus concepts of derivatives and anti-derivatives. Future extensions of the software GeoGebra will include more symbolic features of computer algebra systems which will further increase possible applications for the learning and teaching of calculus.

In addition, we emphasized the increasingly important role of free open-source software packages for mathematics teaching world-wide. Open-source packages do not only offer opportunities for teachers and students to use them both at home and in the classroom without any restriction, but they also provide a means for developing support and user communities reaching across borders. Such collaboration also contributes to the equal access to technological resources and democratization of mathematics learning and teaching.

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