Circles & Angles:

The following series of worksheets were written to help students discover some relationships with angles that are created by tangents, chords, and secants in circles.

Lesson: Central and Inscribed Angles
Grade Level: Secondary Level (Geometry)
Sunshine State Standard: MA.A.1.4.2, MA.A.2.4.2, MA.B.1.4.2, MA.B.2.4.1, MA.B.4.4.1, MA.C.1.4.1, MA.C.2.4.1, MA.D.1.4.1

Materials:

- Students: The use of GeoGebra dynamic worksheets
- Teachers: Projection of GeoGebra dynamic worksheets

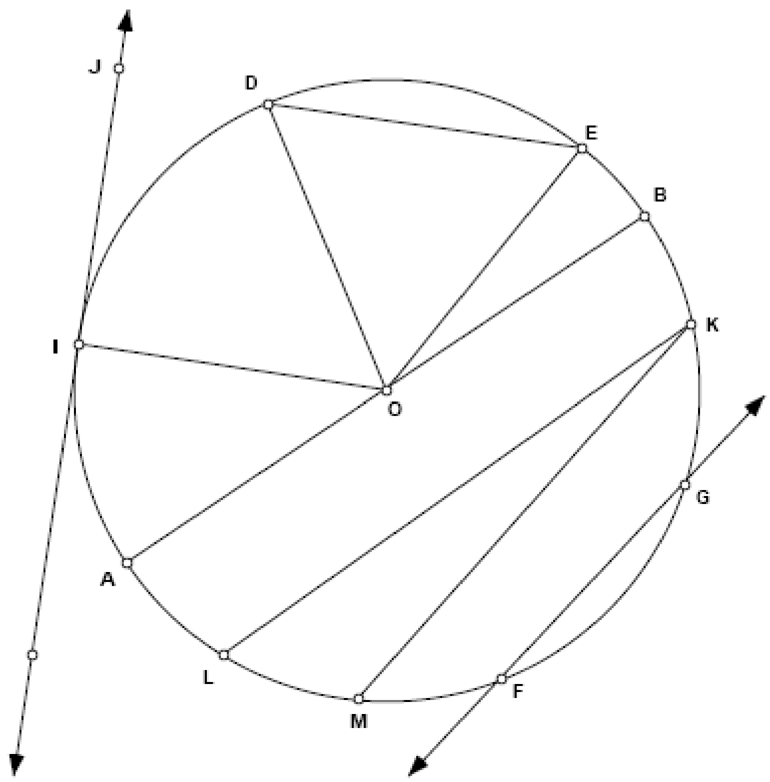
Objectives:

1. Students will discover properties of an angle inscribed in a circle
2. Students will discover properties of the interior angles of a cyclic quadrilateral.
3. Students will discover properties of angles that are formed when two chords of a circle intersect.
4. Students will discover properties of angles formed by two intersecting secants of a circle.
5. Students will be able to find the center of a circle.

Vocabulary: tangent line (segment), secant line (segment), central angle, inscribed angle, cyclic quadrilateral, chord, diameter, radius, right angle, arc (major and minor), intercepted arc, measure (angles and arcs), center, perpendicular bisector

Lesson Plan: (These lessons should be taught during a unit on circles. Each separate dynamic worksheet topic will probably take one class setting, approximately 50 minutes.)

-To start a discussion about circles it may be a good idea to discuss some definition of terms that deal with a circle. Displaying a image similar to one shown below and having students recall what different geometrical figures are called may be one way of starting a lesson on circles.



The following descriptions should be mentioned if using the image above:

The above circle is called “Circle O”, because the center of the circle is the point “O”.

DE is a chord - ***segment whose endpoints lie on the circle.***

AB is a diameter - ***longest chord, or chord containing the center.***

FG is a secant - ***a line containing a chord.***

IJ is a tangent - ***a line intersecting a circle in one point (I is the point of tangency).***

∠*LKM* is an inscribed angle - ***an angle whose vertex is on the circle and sides contain chords of the circle.***

∠*DOE* is a central angle - ***an angle whose vertex is the center and whose sides contain radii.***

Arc DE - ***the part of the circle between points D & E inclusive.***
Note: Arc DE may seem a bit ambiguous. If it is the part of the circle between the two points which part is it? There are two parts between points D & E. Tell the students when an arc is named with two points it is called a minor arc, an arc smaller than a semicircle. An arc named with three letters is a major arc, an arc larger than a semicircle.

Take this opportunity to ask the students some questions that may require them to use a little bit of Algebra. For example:

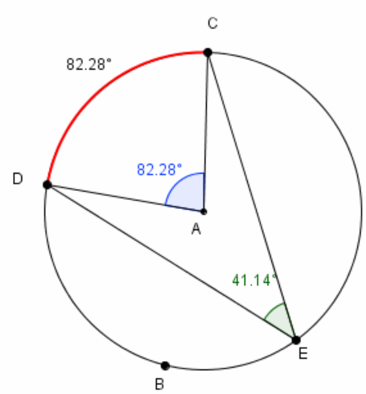
ex. *OB* has a measure of $2x + 3$, what is the measure of *AB*?
A: 4x + 6

-It is now time for students to start their first investigation for this lesson. Students should be given access to the dynamic worksheet, Inscribed_Angle, and be told to follow the directions of the worksheet and come up with a conjecture that deals with a central angle and an inscribed angle.

Inscribed_Angle

Angle CAD is a central angles and Angle CED is inscribed in Circle A. Both angles intercept Arc CD (red arc). Both angles and Arc CD (red arc) have been measured. Use this worksheet to help you answer the questions below.

Note: Try to keep Angle CAD an acute angle, otherwise the measurements get messed up. Why is that?



1. **What relationship does the central angle have with the intercepted arc?**

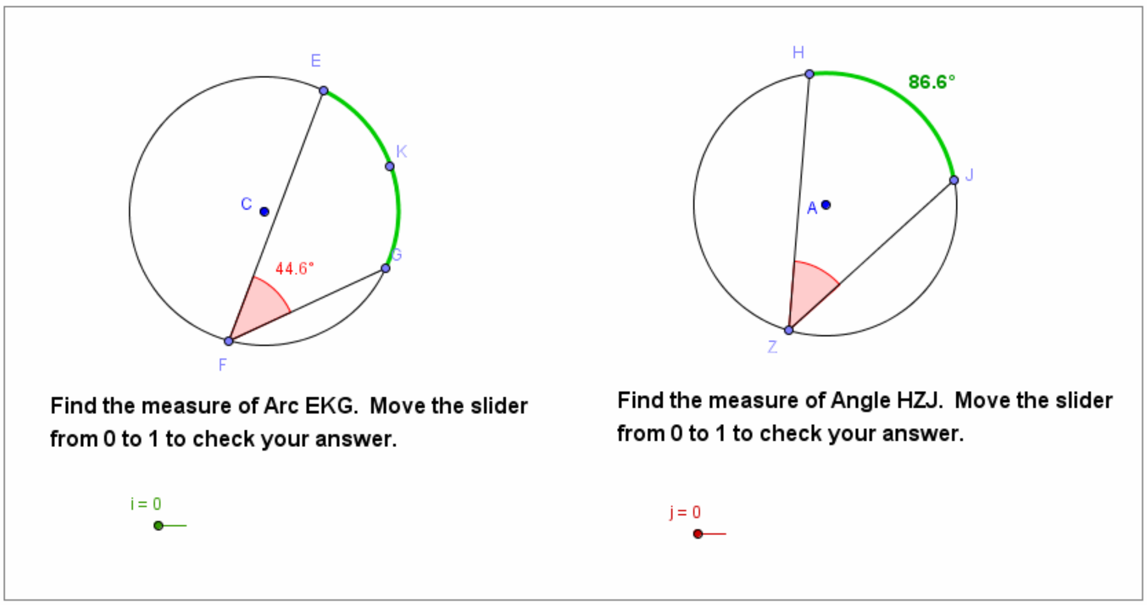
2. **What relationship does the inscribed angle have with the intercepted arc?**

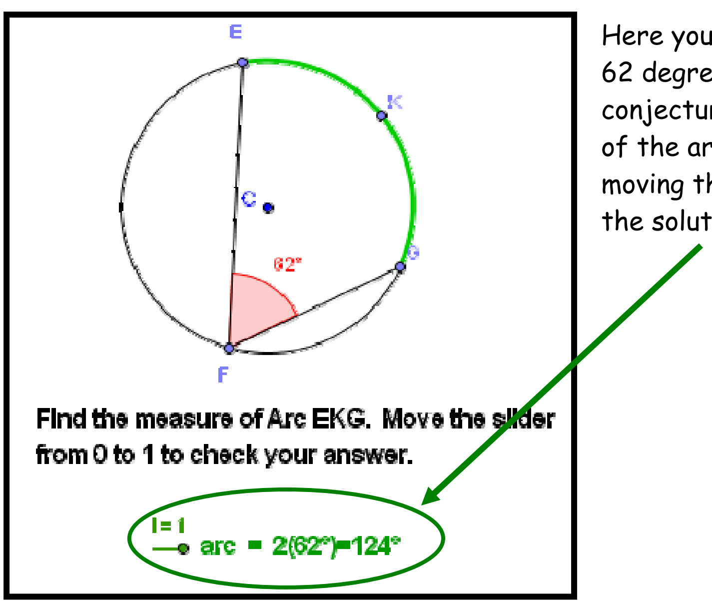
Come up with your conjectures by moving the points
--Point E moves the angle.
(What happens to the measure of the angle when you do this?)
--Points C and D change the size of the angle and intercepted arc.
--Moving Point B will change the size of the Circle A.

Students will hopefully notice that the measure of the intercepted arc is always the same as the measure of the central angle and that the measure of the intercepted arc is double the measure of the inscribed angle. After students have made their conjectures and you have had a class discussion about the dynamic worksheet it is time for the students to use their conjectures. The dynamic worksheet, Inscribed_angle_practice, can be used to have the students create an infinite amount of examples that require the use of their conjectures. The students can also check their answers using the slider below the problem. The text that appears is dynamic text and will recalculate the measure as the arcs and angles change size.

Inscribed_angle_practice

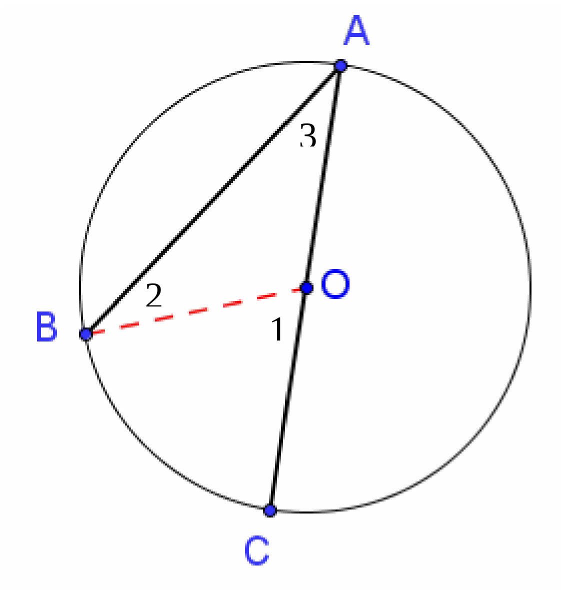
Use the blue points to change the size of the red angle and the green arc. For each problem do 4 - 7 examples. As you complete each problem check your answer by moving the slider.





Here you can see that the inscribed angle is 62 degrees. The students should use their conjecture and figure out that the measure of the arc is 2(62) = 124 degrees. By moving the slider from 0 to 1 you can see the solution.

Below is a proof of the conjecture concerning an angle inscribed in a circle.

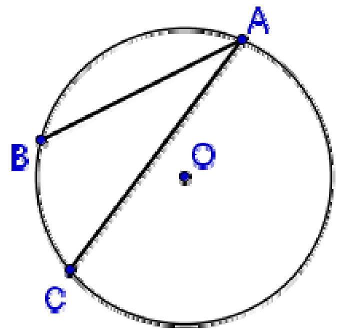


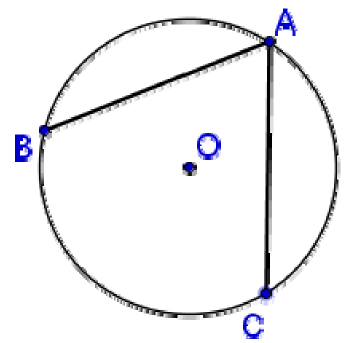
Given: figure as shown above

Prove: $m\angle 3 = \frac{1}{2} m(Arc\ BC)$, **Note: First draw auxiliary line segment "BO".**

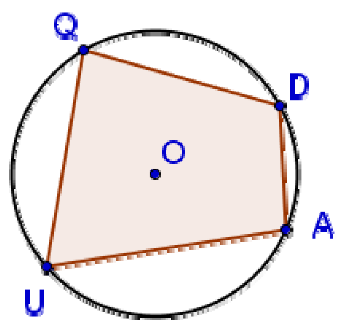
<u>Statements</u>	<u>Reasons</u>
1.) $m\angle 1 = m\angle 2 + m\angle 3$	1.) Measure of an exterior angle is equal to the sum of the remote interior angles.
2.) $\overline{AO} \cong \overline{BO}$	2.) All radii of a circle are congruent.
3.) $m\angle 2 \cong m\angle 3$	3.) If 2 sides of a triangle are congruent, then opposite angles are congruent.
4.) $m\angle 2 \cong m\angle 3 + m\angle 3$	4.) Substitution: (Steps 1 & 3)
5.) $m\angle 2 \cong 2(m\angle 3)$	5.) Arithmetic
6.) $\frac{1}{2} m\angle 1 = m\angle 3$	6.) Multiplication property of equality
7.) $m\angle 1 = m(Arc\ BC)$	7.) Definition of central angle
8.) $\frac{1}{2} m(Arc\ BC) = m\angle 3$	8.) Substitution: (Steps 6 & 7), $m\angle 3$ is $\angle A$.

*Note: The above proof only deals with one case of an angle inscribed in a circle, the case when one ray goes through the center of the circle. There are two other cases that should be discussed. The cases are:





-Now that the students have an understanding of inscribed angles they can start an investigation of inscribed polygons. The first investigation will be that of a cyclic quadrilateral, a quadrilateral that is inscribed in a circle.



The investigation will have the students focus on the quadrilateral's interior angles. Have students use the dynamic worksheet, Inscribed_Quadrilateral, to come up with their conjecture. The first question the students are asked to answer is:

1. *What relationship do the angles of the Quadrilateral have with each other?*

Most students will probably say that all the angles add up to 360 degrees, while being correct, this is not the observation this worksheet is trying to elicit. After moving over Hint_1, the students will be told to:

- *Hint_1: Compare the angles by investigating the sums of angles.*

This hint will probably make students feel more confident about their conjecture that the angles sum to 360 degrees, however after the students move over Hint_2, they will see that their focus should be on the sum of the opposite angles.

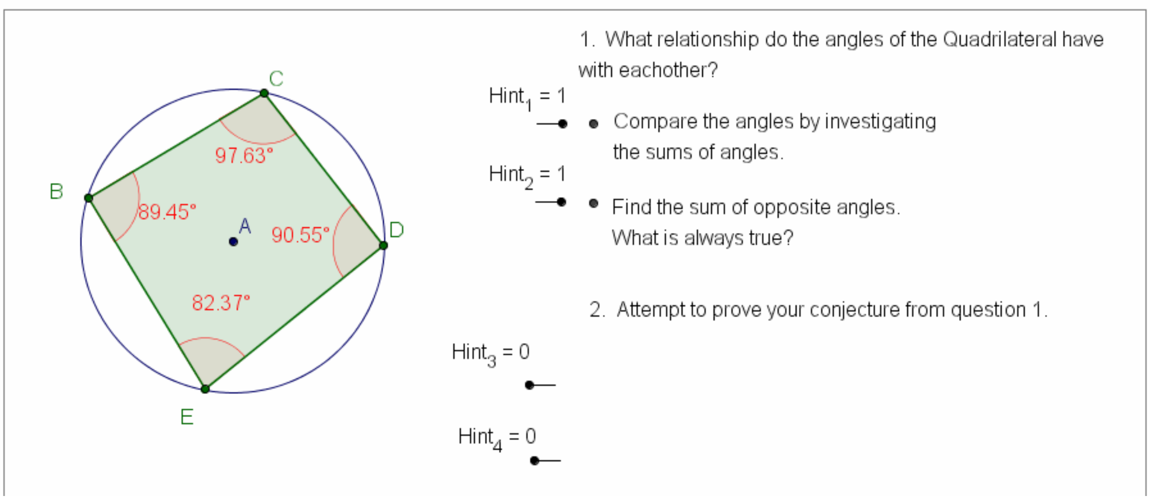
- *Hint_2: Find the sum of opposite angles. What is always true?*

Students should be encouraged to investigate by finding the sum of the opposite angles. By moving the points around the students have an opportunity to check the sums of the opposite angles for many types of cyclic quadrilaterals.

Inscribed_Quadrilateral

Quadrilateral BEDC is inscribed in Circle A. Moving the vertices of the quadrilateral will change it's size and shape, (only Point B controls the size). Use this worksheet to help you answer the questions below.

Note: Make sure to keep the quadrilateral convex. To see the hints move the hint the hint sliders from 0 to 1. Use the hints in order and only use them after you have investigated the question on your own.



Hopefully the students will discover that the opposite angles in an inscribed quadrilateral always sum to 180 degrees.

The second question…

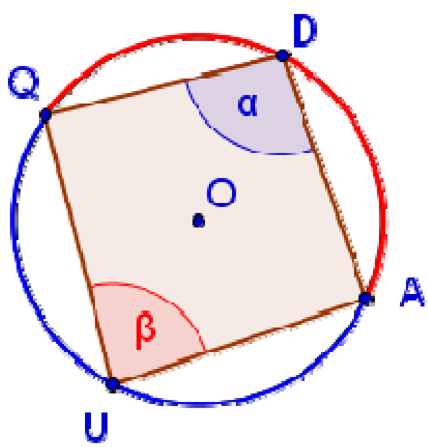
- *Attempt to prove your conjecture from question 1.*

…and the hints…

- *Think about inscribed angles and intercepted arcs.*

- *When you move one vertex what happens to the measure of the angle? Why?*

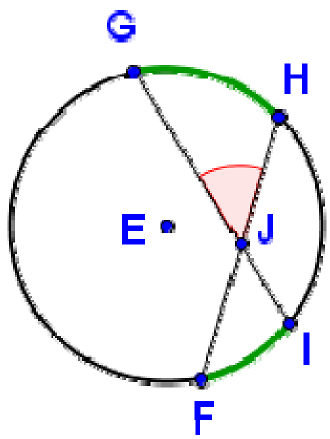
…ask the students to think about a way of proving their conjecture from question 1. This can be shown to the students by explaining the following:



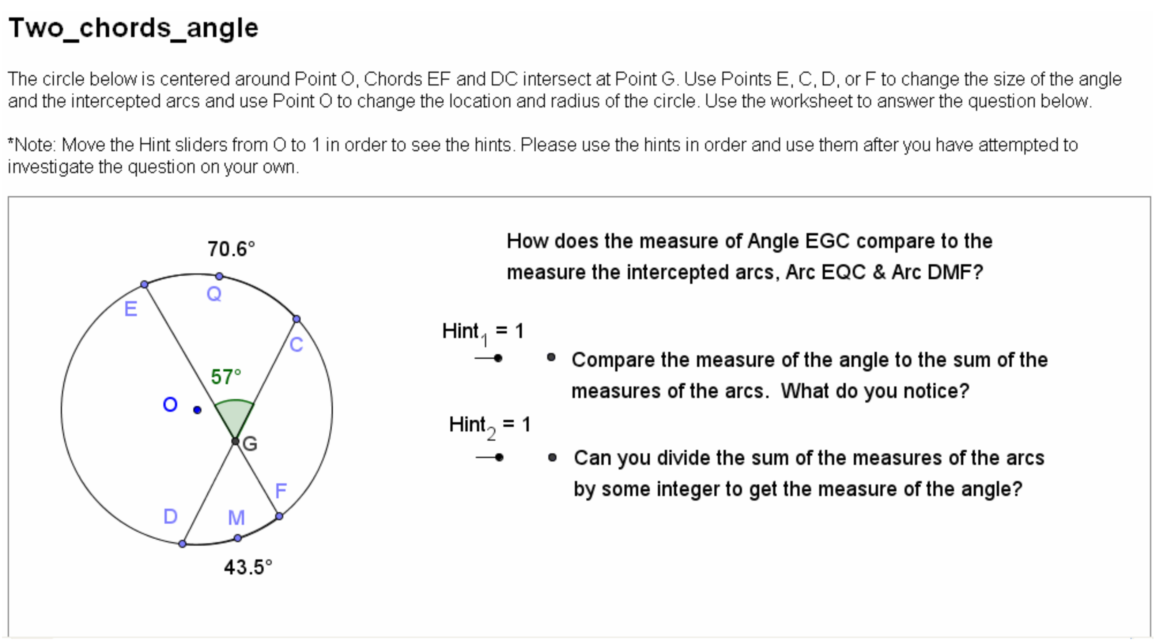
Students should notice that *Angle alpha* intercepts *Arc QUA* and *Angle beta* intercepts *Arc QDA*. We can see that *m(Arc QUA) + m(Arc QDA) = 360,* Since the inscribed angles equals half the measure of the intercepted arcs then,

$$m\angle\alpha + m\angle\beta = 180°$$

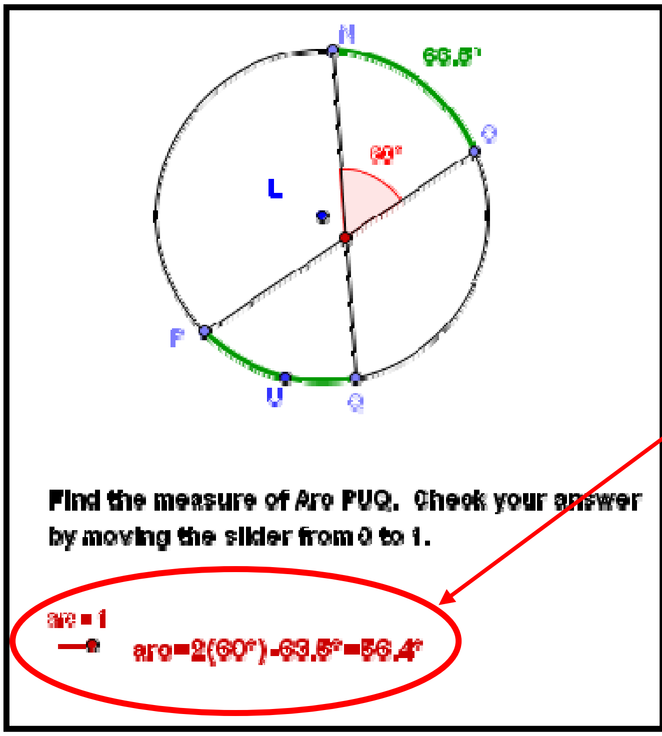
-We will now take our investigations to try to find the measure of angles that are formed by chords or secants to the circle. Our first investigation will deal with two intersecting chords. The goal is to be able to find the measure of one of the angles that is formed.



Have the students use the dynamic worksheet, Two_chords_angles, to come up with a conjecture about the angle formed by two intersecting chords and the arcs the chords intercept.

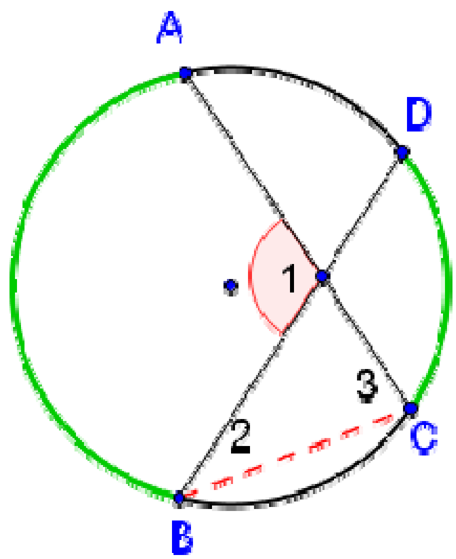


With the use of the worksheet and hints, the students should be able to discover that the angle formed by two intersecting chords is always half the sum of the intercepted arcs. After students have made their conjectures and you have had a class discussion about the dynamic worksheet it is time for the students to use their conjectures. The dynamic worksheet, Two_chords_angles_practice, can be used to have the students create an infinite amount of examples that require the use of their conjectures. The students can also check their answers using the slider below the problem. The text that appears is dynamic text and will recalculate the measure as the arcs and angles change size.



Here you can see that the angle is 60 degrees and one of the arcs is 63.5 degrees. The students should use their conjecture and figure out that the measure of the other arc is 2(60)-63.5 = 56.4 degrees. By moving the slider from 0 to 1 you can see the solution.

Below is a proof of the conjecture concerning the angle formed when two chords intersect in a circle.

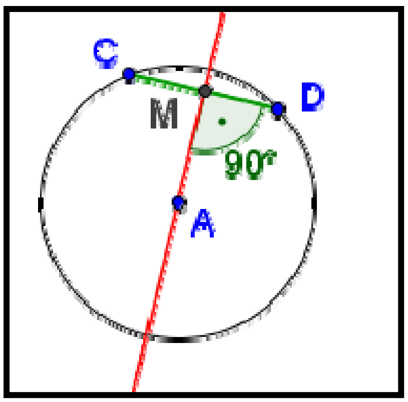


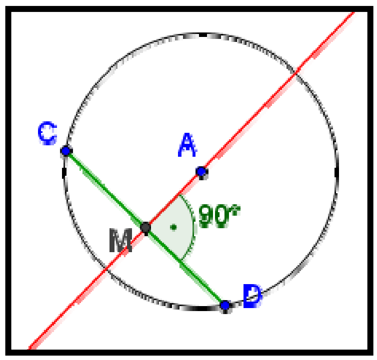
Drawing the auxiliary line as shown and numbering the angles for easy labeling, we are ready to try to prove our conjecture. You may want to see if any of your students can come up with the proof of this property after seeing the auxiliary line.

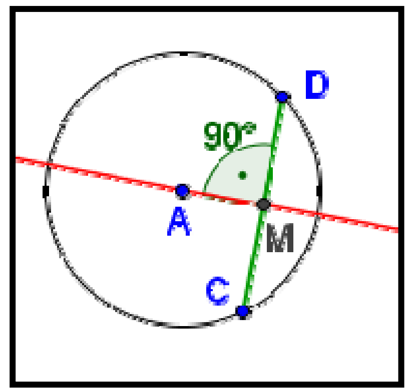
Statements	Reasons
1.) $m\angle 1 = m\angle 2 + m\angle 3$	1.) In a triangle an exterior angle is equal to the sum of the measures of the two remote interior angles.
2.) $m\angle 2 = \frac{1}{2}m(Arc\ DC)$ $m\angle 3 = \frac{1}{2}m(Arc\ AB)$	2.) Inscribed angle is half the intercepted arc.
3.) $m\angle 1 = \frac{1}{2}m(Arc\ DC) + \frac{1}{2}m(Arc\ AB)$	3.) Substitution (Steps 1 & 2)
4.) $m\angle 1 = \frac{1}{2}[m(Arc\ DC) + m(Arc\ AB)]$	4.) Factor out the $\frac{1}{2}$ from the right side of the equation

-There is a similar dynamic worksheet, Two_secants_angles, which deals with two secants that intersect at a point outside the circle. There is also a practice dynamic worksheet, Two_secants_angles_practice, which will allow the students to use the conjectures they came up with while investigating the worksheet.

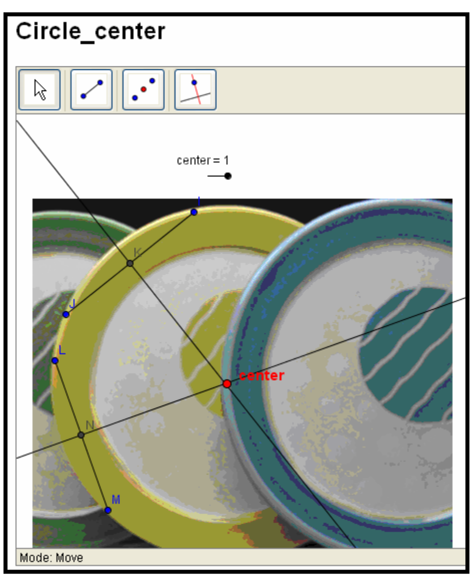
-Changing gears a little bit from angles, our last investigation is going to deal with what happens when a chord is perpendicularly bisected. The students will first be asked to use the dynamic worksheet, bisecting_chords, to make a conjecture.







Students will notice that the perpendicular bisector of any chord always goes through the center of the circle.



The next part of this lesson will be to have the students use the dynamic worksheet, Circle_center, to try to find the center of the yellow plate. In this dynamic worksheet the students have the ability of using four tools: the pointer, chord, midpoint, and perpendicular line tools. After the students believe they have found the center they can move the slider from 0 to 1 to check their estimation.